FoLiATioNS

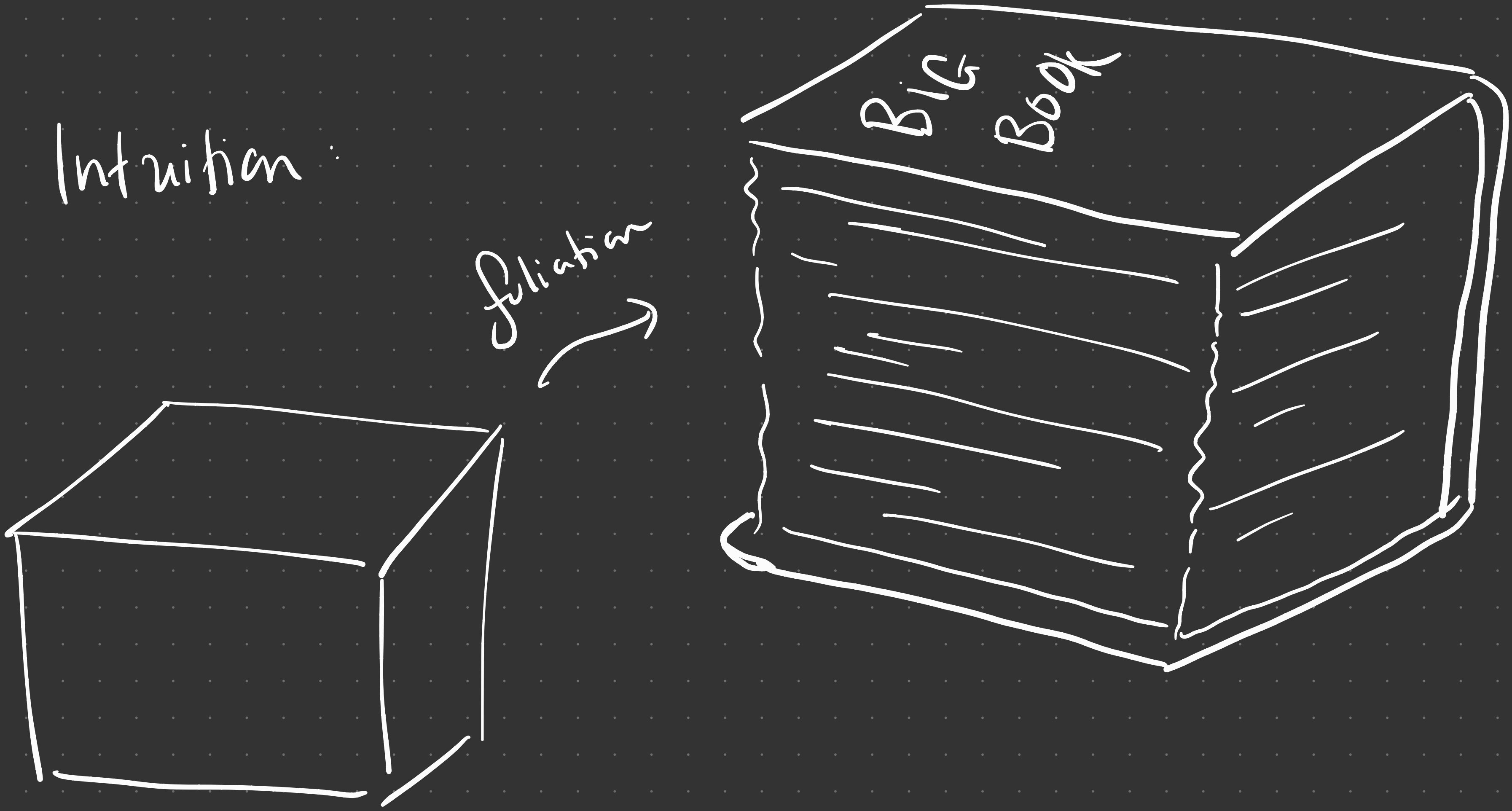
Outline

1. Def^n + Examples
2. Codim 1 in 3 mflds
3. Taut / Transverse
4. MISC.

C. MONROE | [UT - L-Space Sem] 10/29/21

① Def^n + Examples

Intuition:



Def^{n}: A foliation $\mathcal{F}$ on

an n-mfld M $(\partial M = \emptyset)$

is a disjoint union $\coprod_{\lambda \in \Lambda} L_\lambda$

of connected k-mflds, for some

$0 \leq k < n$; (leaves)

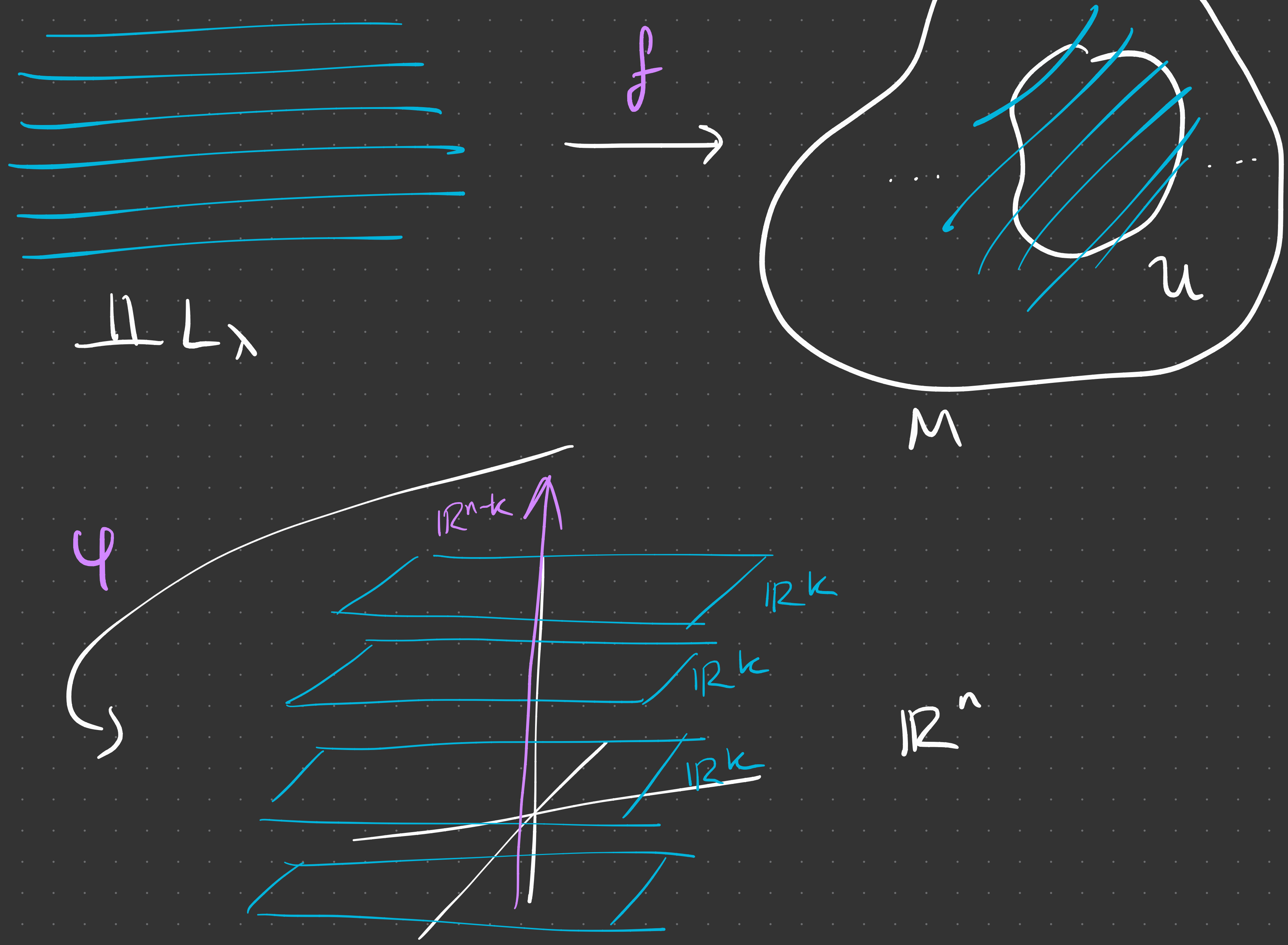
and a continuous bijection $f : \coprod_{\lambda \in \Lambda} L_\lambda \to M$

where M is covered by coordinate charts $\varphi : \xrightarrow{\cong} \mathbb{R}^n$

s.t. $\forall \lambda \in \Lambda$ $\quad \varphi(f(L_\lambda) \cap U) = \mathbb{R}^k \times X_\lambda$

for some $X_\lambda \subset \mathbb{R}^{n-k}$ (maps)

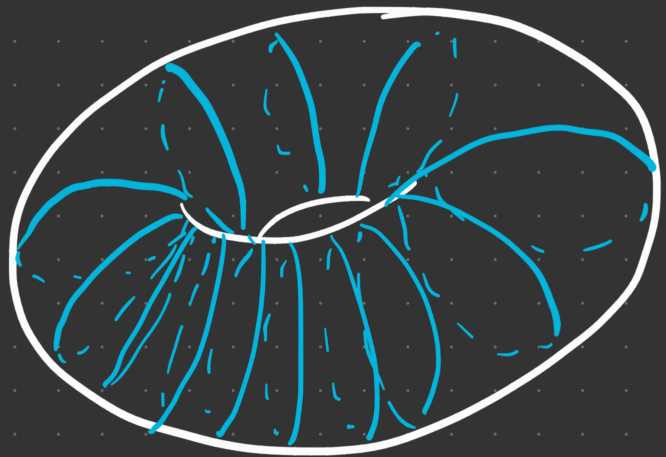
CARTOON



$\coprod L_\lambda$
f
$\mathcal{U}$
M
ψ
$\mathbb{R}^{n-k}$
$\mathbb{R}^k$
$\mathbb{R}^k$
$\mathbb{R}^k$
$\mathbb{R}^n$

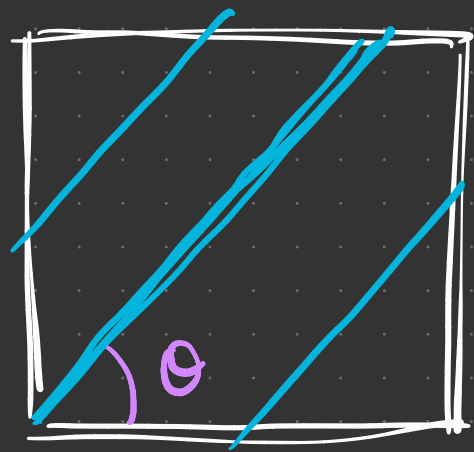
EXAMPLES!

woo!



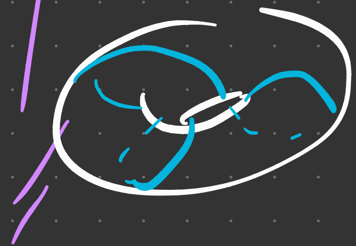
$S^1 \times S^1$

$\mathcal{F}$: $S^1 \times \{k\}$, $k \in [0,1]$

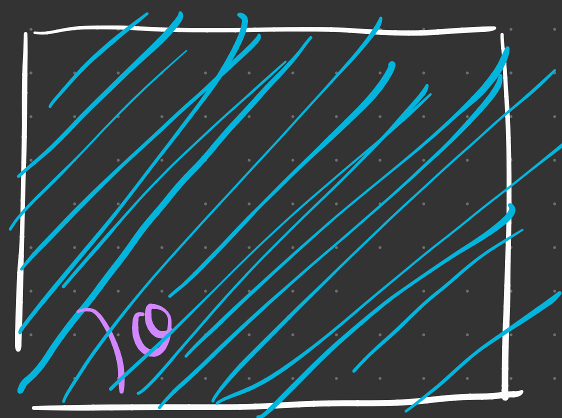


· if $\theta \in \mathbb{Q}$

· if $\theta \notin \mathbb{Q}$



$[0,1] \times [0,1]$



More examples:

- A Seifert Fibered Space has a codim 2 foliation where the leaves are circles

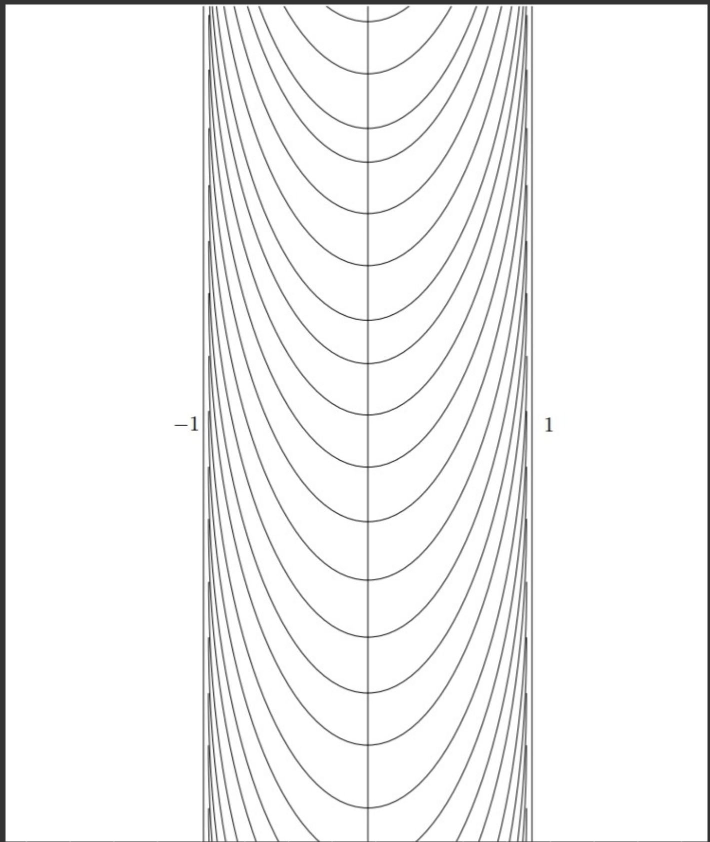
- Thm: If M is a closed 3-mfld w/ a foliation w/ leaves homeo to $\mathbb{R}^2$, then $M \cong T^3$

- A fiber bundle $F \to M$ gives a foliation of M

$$F \to M \downarrow B$$

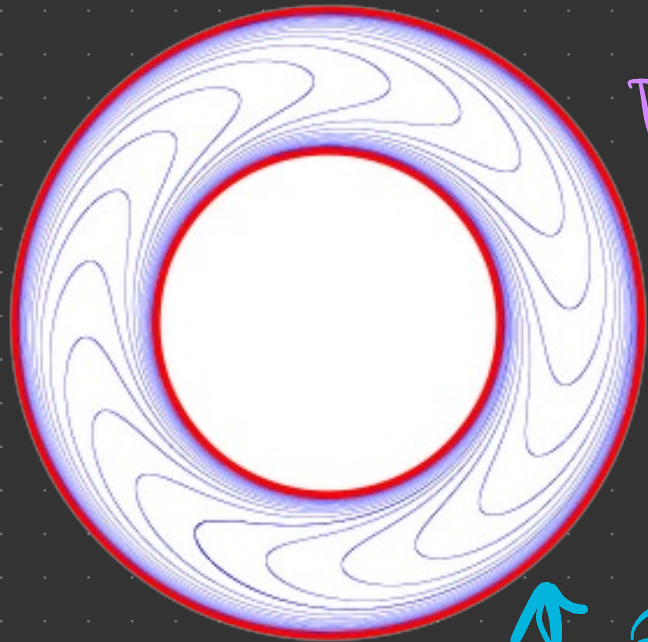
where the leaves are the fibers

Important Example: REEB FOLIATION of



→ rotate about y-axis to get foliation of $D^2 \times \mathbb{R}$

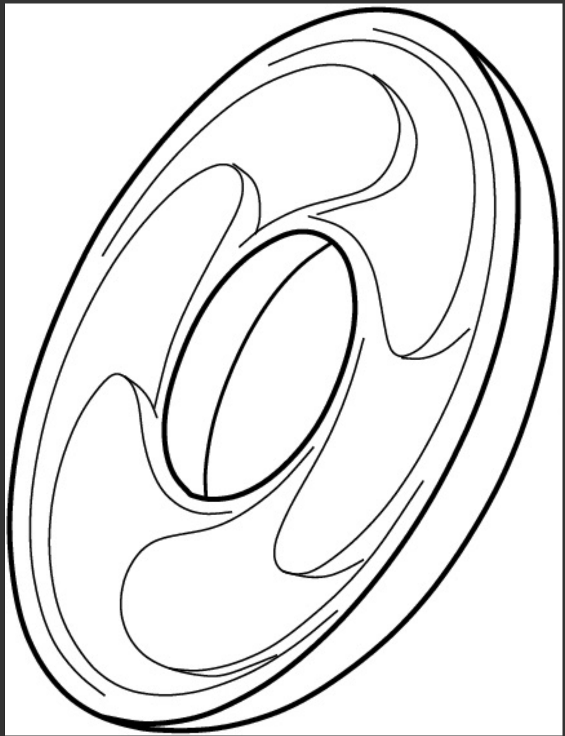
quotient



$D^2 \times S^1$

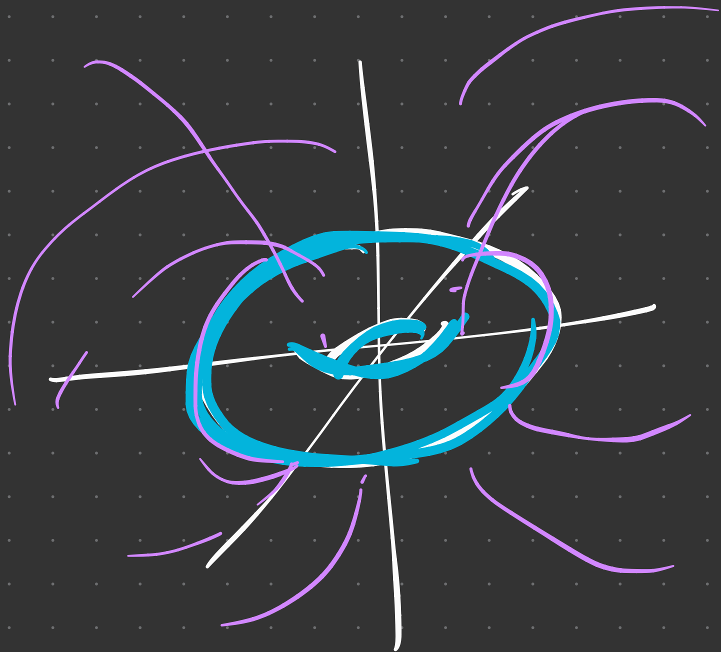
$\partial = T^2$

Foliation of $[-1, 1] \times \mathbb{R}$ w/ leaves homeo to $\mathbb{R}$



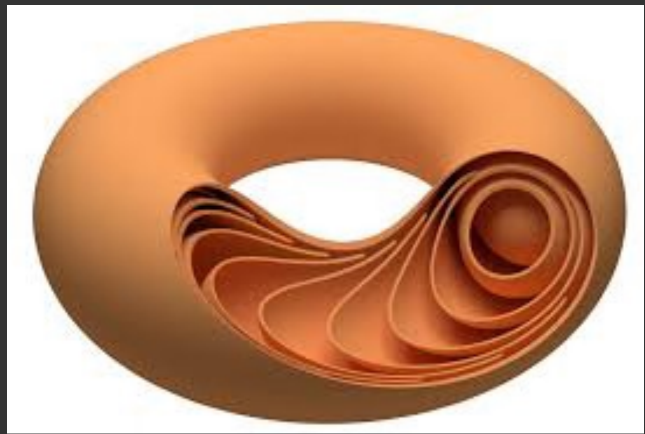
Note: Since S^3 has a splitting as 2 solid tori, the Reeb foliation gives us a codim 1 - foliation.

[CAMERON STORY HERE]



② Codim 1 foliations of 3-mflds

Thm: Every closed 3-mfld has a codim-1 foliation.



Sketch of Pf:

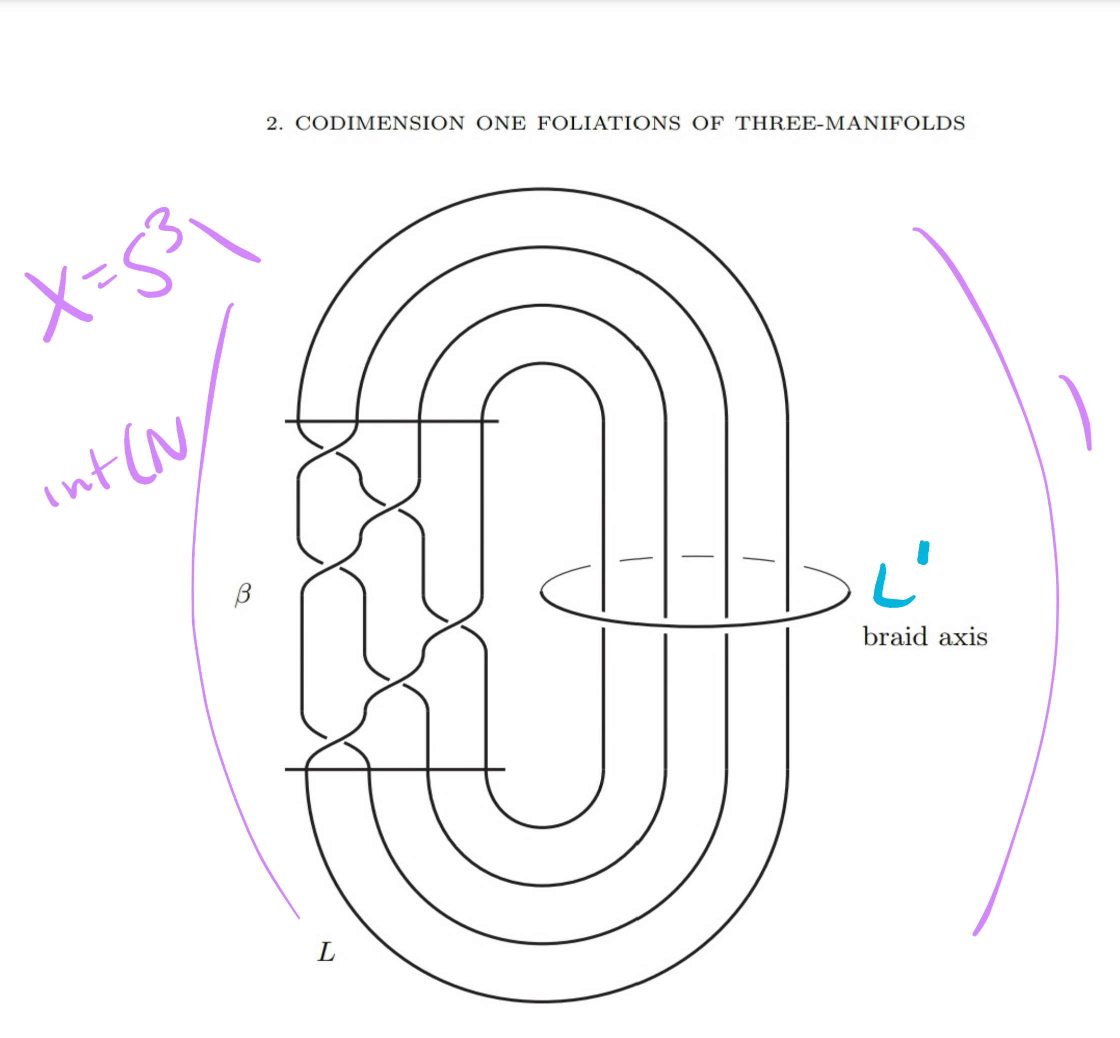
Need 2 facts:

1. Thm[Likorish-Wallace]: Every closed 3-mfld can be obtained by Dehn surgery on a link in S^3

2. Thm[Alexander]: Every link in S^3 is the closure of some braid

IDEA: • M is surgery on a link L, and L is the closure of some braid β. $L' = L \amalg$ Braid Axis

$X = S^3 \setminus \operatorname{int}(N$



• X is an F-bundle over S^1, F is an n-punctured disk

• There is a codim-1 foliation of X given by F; $\partial X = \coprod_{i=0}^{m} T_i$, m # components of L

• Spin the leaves and T_i to get a new foliation F'

$M = X \cup (\amalg V_i)$ solid tori w/ Reeb foliations

NOTICE: There's a sence in which

- codim 1 foliations of closed 3-mflds are not special
- because Reeb foliations are

ALSO: If a mfld M has a (cooriented) codim 1 foliation, then $F: M \to M$ def'd by pushing off the leaves

↳ F has no fixed pts BUT is isotopic to id

↳ Lefschetz fixed pt Thm $\Rightarrow \chi(M) = 0$

GOAL: Finding codim 1 foliations shows your $\chi(M) = 0$!

(But then)

Thm [Thurston]: A closed n-mfld M has

codim 1 foliation iff $\chi(M) = 0$.

Speaking of cool theorems that use foliations to give you important topological information...

Reeb's Stability Thm: If one leaf of a codim-1 foliation is closed and has finite π_1, then all the leaves are closed and have finite π_1.

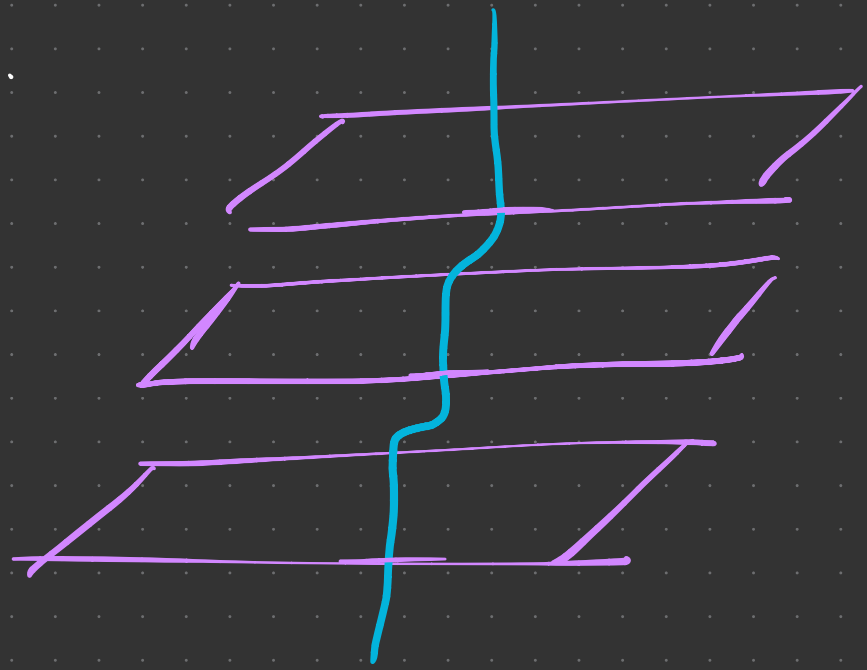
Special Case: Let M be a closed 3-mfld w/ a foliation $\mathcal{F}$ w/ a leaf homeo to S^2 or $\mathbb{R}P^2$.

Then $M \cong S^2 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$

3 Tant | Transverce

Def^n: Let $\mathcal{F}$ be a codim 1 foliation on M.

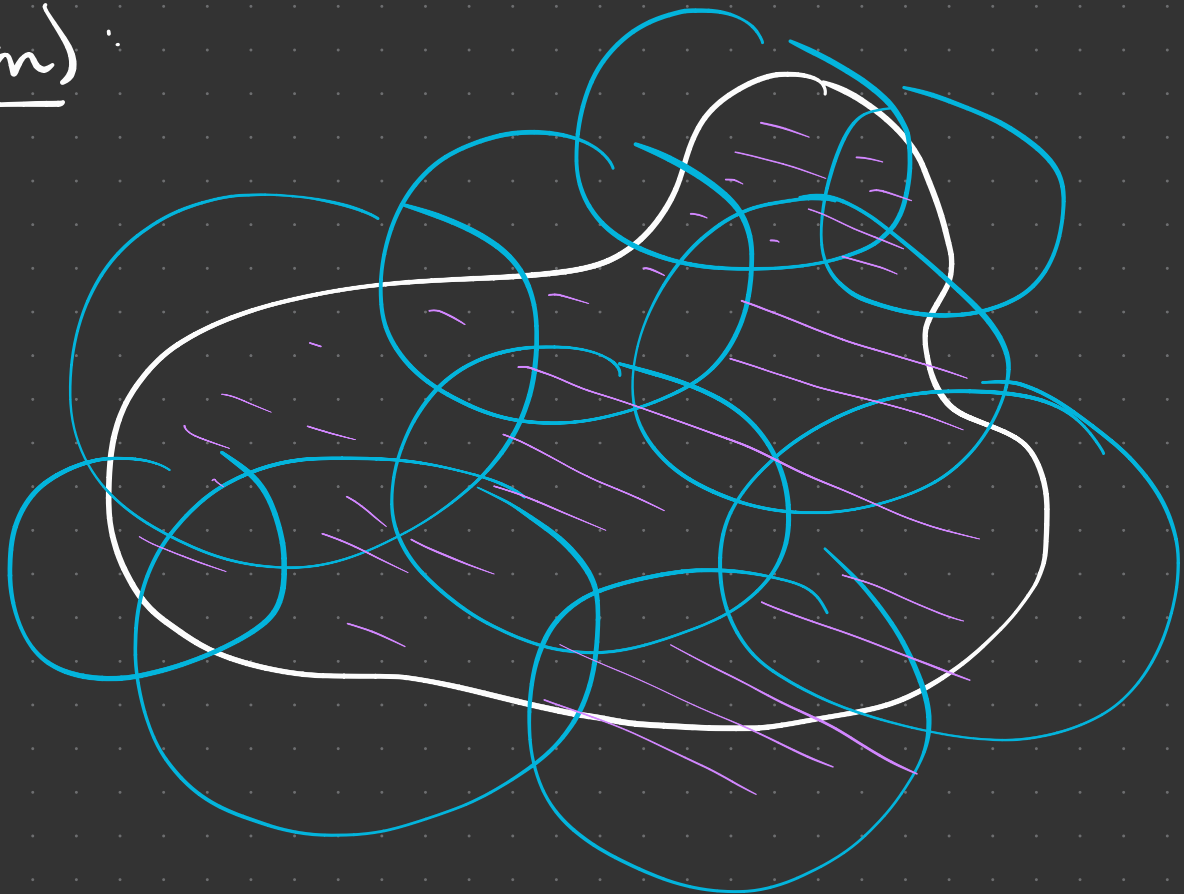
A transverce loop in M is a loop which is transverce to $\mathcal{F}$.



Lemma: If M is compact then there is a transverse loop in M.

Pf (by exhaustion):

"Proceed transversely."



Novikov's Thm: Let $\mathcal{F}$ be a Reebless foliation on a closed 3-mfld not homeo to $S^2 \times S^1$ nor $\mathbb{RP}^3 \# \mathbb{RP}^3$. Then

(1) for any leaf L of $\mathcal{F}$, $\pi_1(L) \hookrightarrow \pi_1(M)$

(2) every transverse loop is essential

REM: "Reebless" as a condition makes codim 1 foliations once again interesting!

<u>COR</u>: Say $\mathcal{F}$ has a transverse loop γ.

γ^n is also transverse $\forall n \geq 1$.

$\Rightarrow [\gamma]$ has inf. order in $\pi_1(M) \Rightarrow \pi_1(M)$ infinite

$\Rightarrow \tilde{M}$ is non-compact $\Rightarrow$ by (i) $\pi_1(\tilde{L}) = 1$

$\Rightarrow$ Reeb Stability + our choice of M, all $\tilde{L} = \mathbb{R}^2$

$\Rightarrow \tilde{M} = \mathbb{R}^3 \iff M$ is irreducible

$+$

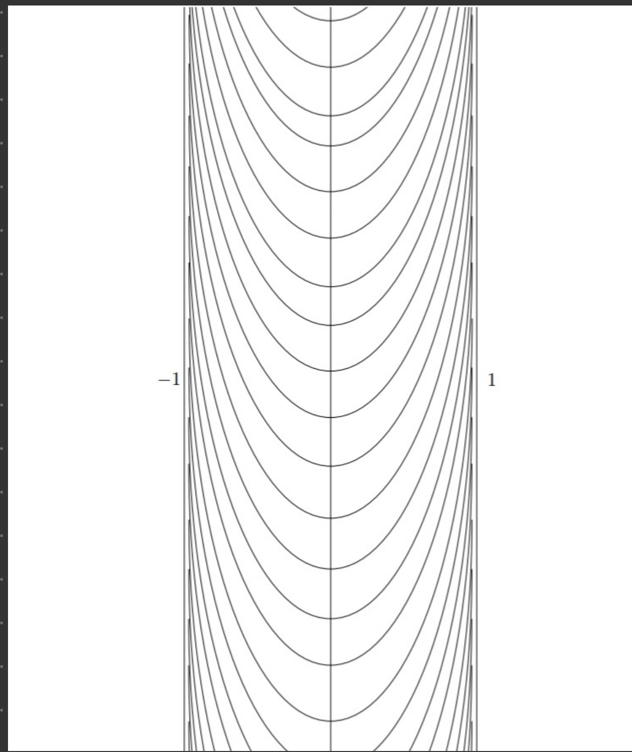
$\pi_1(M)$ is inf.

[Palmeira's Thm]

Def^n: Let $\mathcal{F}$ be a foliation of a closed 3-mfld M.

$\mathcal{F}$ is **taut** if $\mathcal{F}$ has a transverse loop γ s.t. $\forall$ leaves L, $L \cap \gamma \neq \emptyset$.

REM: If we have Reeb foliations, we can't have a transverse loop.



TAUT $\rightarrow$ Reebless

TAUT $\longrightarrow$ Reebless

CONVERSE FALSE : T_0, $\partial T_0 = S^1$

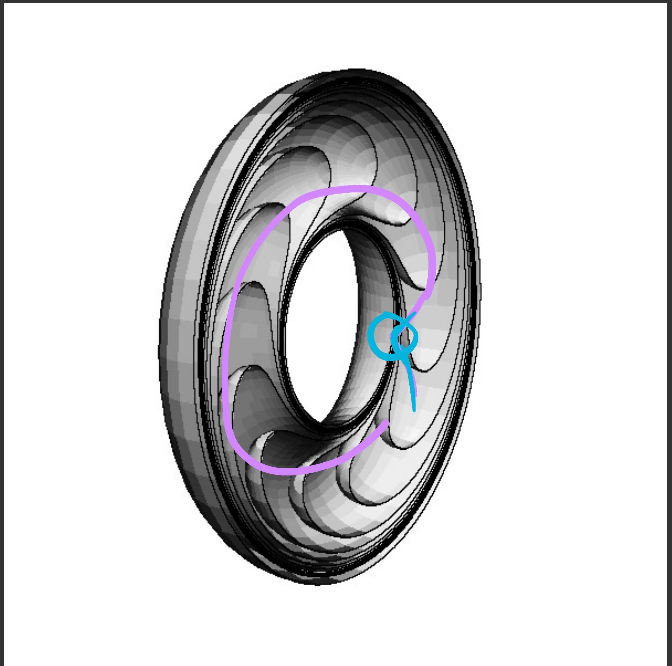
$X = T_0 \times S^1$; Spin $T_0 \times \{pt\}$ and ∂X

gives a foliation $\mathcal{F}$ of X.

$M = X' \cup_\partial X$ $\rightarrow \mathcal{F}, \mathcal{F}'$ give a foliation of M

$T = \partial X = \partial X' \subset M$ is Reebless, but not Taut

Thm [Goodman]: If a foliation $\mathcal{F}$ on a closed orientable 3-mfld is <u>not</u> taut, $\mathcal{F}$ has a torus leaf



4 Misc.

Defn: If $\mathcal{F}$ has a codim 1 foliation on a closed n-mfld, we say $\mathcal{F}$ is co-orientable if there is a consistent transverse orientation to the leaves of $\mathcal{F}$.

[Thm]: Let M be a closed n-mfld.

If M has a codim-1 foliation $\mathcal{F}$,

then $\chi(M) = 0$.

Pf: The foliations from our braid/surgery construction

are coorientable $\Longrightarrow$

in a double cover $\tilde{M} \to M$, $\tilde{\mathcal{F}}$ lifts to a coorientable

nonvanishing vector field $\Rightarrow$ $\chi(\tilde{M}) = 2\chi(M) = 0$ □

(1/2 - L-space)

CONJECTURE: If M is a closed prime 3-mfld, then M has a CTF iff $\pi_1 M$ is left-orderable.

↪ Thm [Gabai]: If M is a closed prime 3-mfld, with $H_1(M)$ inf, M has a CTF.

(solved for $H_1(M)$ inf).

That's all