Contact structures and symplectic fillings of Seifert fibered spaces

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I. Why should we care about contact topology?

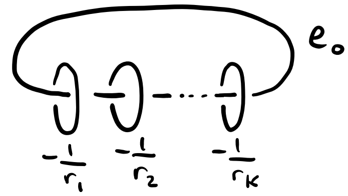
One reason: We can characterize certain Seifert fibered QHS's as L-spaces via contact structures:

Theorem (Lisca-Stipsicz). If M is an oriented QHS which is Seifert fibered over S^2, TFAE:

(1) M is an L-space
(2) Either M or $-M$ does not admit a positive transverse contact structure
(3) M carries no transverse foliations
(4) M carries no taut foliations

$$-\alpha \qquad \alpha \wedge d\alpha < 0$$
$$\uparrow$$
$$-\alpha \wedge d(-\alpha) = \alpha \wedge d\alpha < 0$$
$$\underbrace{-(\alpha \wedge d\alpha)}_{\beta \wedge d\beta} > 0$$

An oriented QHS M which admits a Seifert fibering over S^2 is (orientation-preserving) diffeomorphic to the 3-mfld

$M(e_0; r_1, r_2, \ldots, r_k) =$ 

where $e_0 \in \mathbb{Z}$, $r_i \in \mathbb{Q} \cap (0,1)$ and $r_1 \geq r_2 \geq \cdots \geq r_k$.

By Lisca-Matić, an oriented SF QHS

$$M = M(e_0; r_1, r_2, \ldots, r_k)$$

with $r_1 \geq r_2 \geq \cdots \geq r_k$ does not admit a positive transverse contact structure if and only if either:

(i) $e_0(M) \geq 0$, or

(ii) $e_0(M) = -1$ and there are no coprime integers m, a $(m > a)$ such that

$$m r_1 < a < m(1 - r_2)$$

and

$$m r_i < 1 \qquad (i = 3, \ldots, k).$$

This combined with the theorem above gives a characterization of L-spaces among QHS's of the form $M(e_0; r_1, \ldots, r_k)$.

Contact structures

Let Y be a closed, oriented 3-manifold.

Def. A smooth 1-form $\omega \in \Omega^1(Y)$ is a contact form on Y if $\omega \wedge d\omega \neq 0$ everywhere. If $\omega \wedge d\omega > 0$, i.e., $(\omega \wedge d\omega)_p(v_1, v_2, v_3) > 0$ for (v_1, v_2, v_3) a positively-oriented basis of T_pY, then we say ω is a positive contact form.

If ω is a contact form on Y, then we call the plane field

$$\xi := \ker \omega \subset TY$$

a (cooriented) contact structure on Y. If ω is also positive, we call ξ a positive contact structure.

Example. $\mathbb{R}^3_{x,y,z}$ with the standard orientation, and $\alpha_1 = dz + x\,dy$. Note that $d\alpha_1 = dx \wedge dy$ so

$$\begin{aligned}\alpha_1 \wedge d\alpha_1 &= (dz + x\,dy) \wedge (dx \wedge dy)\\ &= dx \wedge dy \wedge x\,dy + dx \wedge dy \wedge dz\\ &= dx \wedge dy \wedge dz\\ &> 0\end{aligned}$$

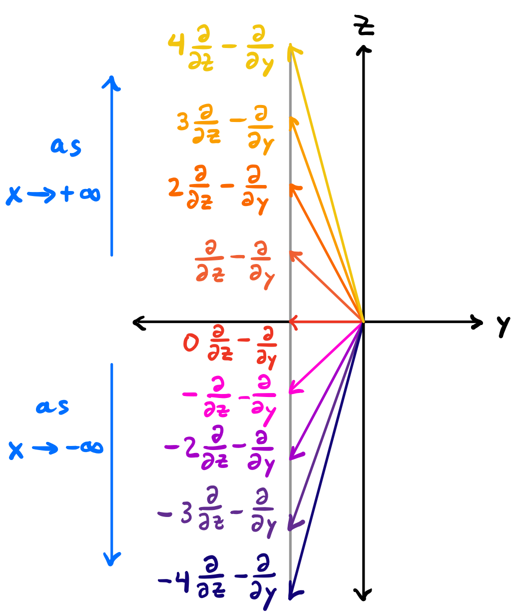
$\Rightarrow$ α_1 is a positive contact form

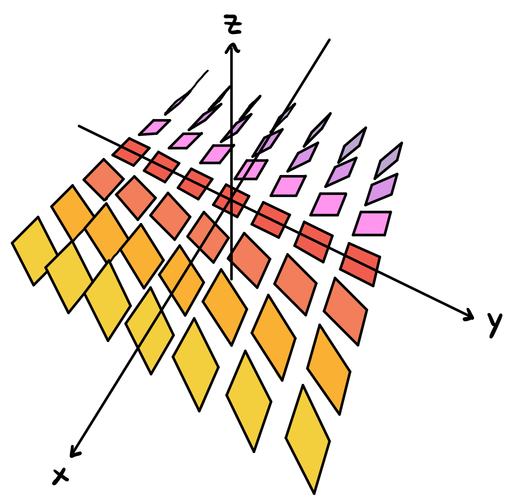
$\Rightarrow$ $\xi_1 = \ker(\alpha_1)$ is a positive contact structure

What does ξ_1 look like?

At a point (x,y,z), the plane ξ_1 is spanned by

$$\left\{ \frac{\partial}{\partial x},\ x\frac{\partial}{\partial z} - \frac{\partial}{\partial y} \right\}.$$





Some authors start with a smooth oriented plane field ξ on an oriented 3-mfld Y and use the fact that, in this scenario, $\xi = \ker(\alpha)$ for some $\alpha \in \Omega^1(Y)$.

Transverse contact structures

Let (M, ξ) be a contact 3-manifold.

Def: A curve $\gamma : S^1 \to M$ is transverse to ξ if for all $x \in S^1$, $d\gamma(T_x S^1)$ is transverse to $\xi_{\gamma(x)}$.

Let $\pi : M \to \Sigma$ be an oriented Seifert fibered 3-manifold.

Def. After identifying each S^1 fiber F_q over $q \in \Sigma$ with a curve $\gamma_q : S^1 \to M$, we say a contact structure ξ on M is transverse to its Seifert fibration if γ_q is transverse to ξ for all $q \in \Sigma$.

Example. Consider the 3-torus $\mathbb{T}^3 = \mathbb{R}^3 / \mathbb{Z}^3$ with orientation induced by the standard orientation on $\mathbb{R}^3$, and let

$$\alpha = \sin(2\pi z)\,dx + \cos(2\pi z)\,dy.$$

Observe that

$$d\alpha = 2\pi \cos(2\pi z)\,dz \wedge dx - 2\pi \sin(2\pi z)\,dz \wedge dy$$

so

$$\alpha \wedge d\alpha = \left(\sin(2\pi z)\,dx\right) \wedge \left(-2\pi \sin(2\pi z)\,dz \wedge dy\right)$$

$$+\ (\cos(2\pi z)dy) \wedge (2\pi\cos(2\pi z)\, dz \wedge dx)$$

$$= 2\pi \sin^2(2\pi z)\ dx \wedge dy \wedge dz$$

$$\quad + 2\pi\cos^2(2\pi z)\ dx \wedge dy \wedge dz$$

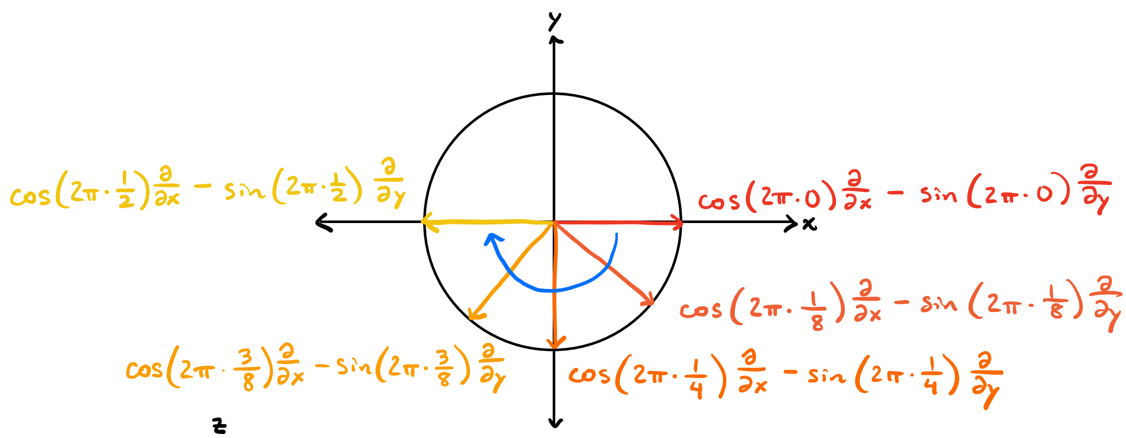
$$= 2\pi\ dx \wedge dy \wedge dz$$

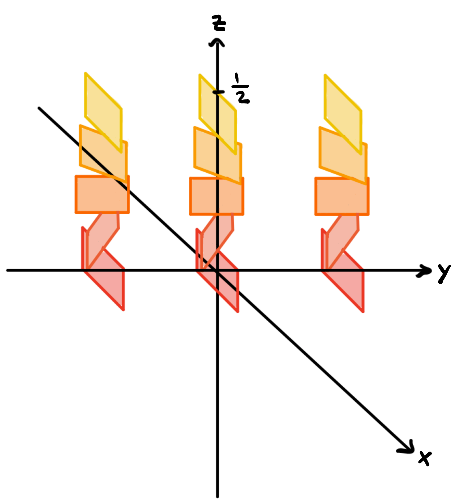
$$> 0.$$

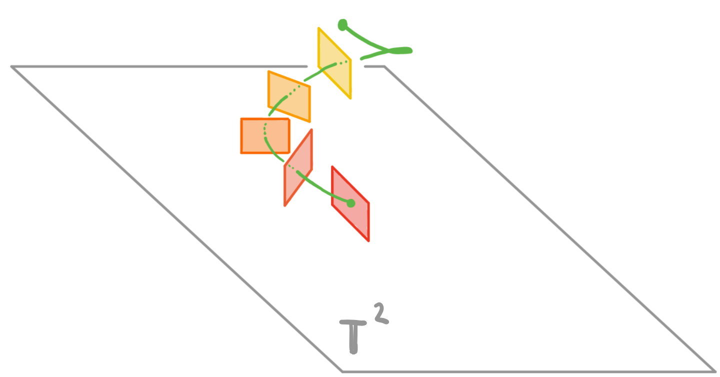
Then $\xi = \ker\alpha$ is a positive contact structure on $\mathbb{T}^2$.

What does it look like? At any point $p = (x, y, z)$, $\ker\alpha$ is spanned by

$$\left\langle \frac{\partial}{\partial z},\ \cos(2\pi z)\frac{\partial}{\partial x} - \sin(2\pi z)\frac{\partial}{\partial y} \right\rangle$$







We also see ξ is transverse to the Seifert fibration of $\mathbb{T}^3$ where the S^1-fibers look like the green one above.

Legendrian and transverse knots

Curves in a contact 3-manifold can illuminate the contact structure. Two kinds of curves are especially important: Legendrian and transverse knots.

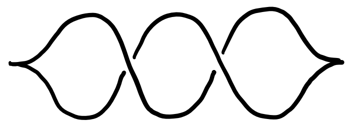
Def. If (M, ξ) is a contact 3-manifold, a knot $\gamma : S^1 \to M$ is Legendrian if $K = \gamma(S^1)$ is always tangent to ξ, i.e., for every $x \in K$, $d\gamma(T_x S^1)$ is contained in $\xi_{\gamma(x)}$.

Consider $\mathbb{R}^3$ with its standard contact structure

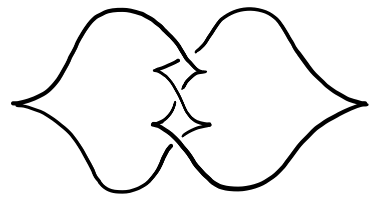
$$\xi = \ker(dz + x\,dy).$$

Let's investigate Legendrian knots in $(\mathbb{R}^3, \xi)$. Let γ be such a knot. Project γ to the yz-plane (this is called a front projection).

Some examples:



Legendrian unknot



Legendrian trefoil

In general:

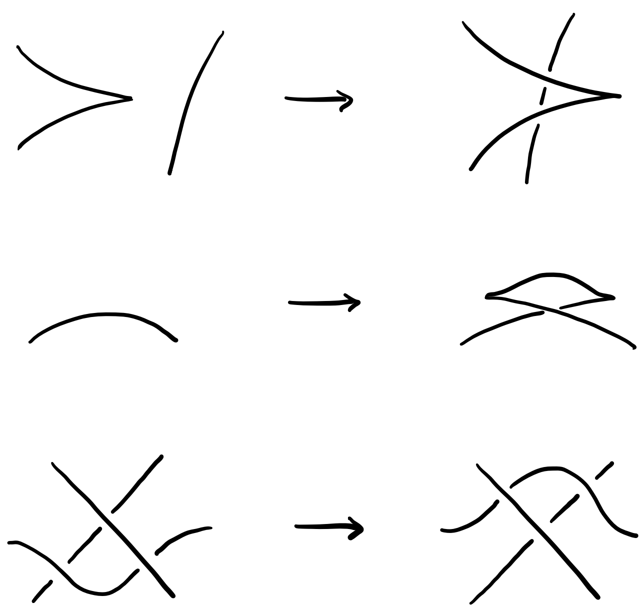
(1) at all the crossings, the strand with the smaller slope lies in front of the strand with larger slope, and

(2) there are no vertical tangencies; instead there are cusps.

These are the only restrictions on a front projection of a Legendrian knot, and any projection satisfying these restrictions is the front projection of a Legendrian knot.

So the study of Legendrian knots in $\mathbb{R}^3$ reduces to the study of their front projections.

In particular, if two Legendrian knots are isotopic, then we can go from one front projection to the other via Legendrian Reidemeister moves:



(Type I)

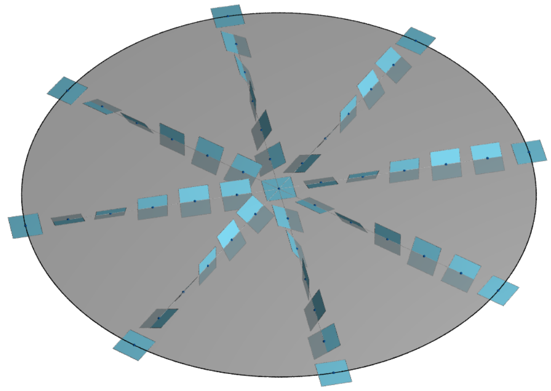
(Type II)

(Type III)

Tight vs. overtwisted contact structures

Let (Y, ξ) be a closed contact 3-manifold.

Def. An embedded disk $D^2 \subset Y$ is an overtwisted disk for ξ if $TD^2|_{\partial D^2} = \xi|_{\partial D^2}$



The contact structure ξ is <u>overtwisted</u> if Y contains an overtwisted disk for ξ, otherwise ξ is called <u>tight</u>.

Overtwisted contact structures are common: any 3-manifold admits an overtwisted contact structure, and on a closed 3-manifold two contact structures are isotopic iff they are homotopic as plane fields. (Eliashberg)

In contrast, tight contact structures are much harder to find in general — their existence is not even known in general.

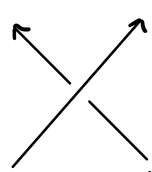
However, tight contact structures are classified up to isotopy on:

- S^3 (Eliashberg?)
- lens spaces (Giroux, Honda)
- S^1-bundles over surfaces (Honda)
- some Seifert fibered spaces (Ghiggini, Ghiggini-Lisca-Stipsicz, Lisca-Stipsicz, Wu)

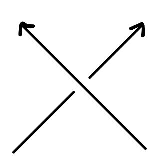
The Bennequin inequality

I promised you that transverse and Legendrian knots shed light on the nature of contact structures. We're almost there!

Recall that the <u>writhe</u> of a knot diagram is the sum over all crossings of a ± 1 at each crossing where

$+1 \leftrightarrow$ 

right-handed crossing

$-1 \leftrightarrow$ 

left-handed crossing

Theorem (The Bennequin Inequality). If γ is a transverse knot with respect to a tight contact structure then

$$w(\gamma) \leq -\chi(\Sigma) \qquad (*)$$

where $w(\gamma)$ is the writhe of the front projection of γ and Σ is any Seifert surface for γ.

Bonus: This gives a lower bound on the genus of a Seifert surface for γ — in general, it's hard to determine the genus, so this is nice.

The punchline is that it turns out that a contact structure is tight if and only if the inequality $(*)$ holds. In particular, this shows the standard contact structure on $\mathbb{R}^3$ is tight (Bennequin showed $(*)$ holds).

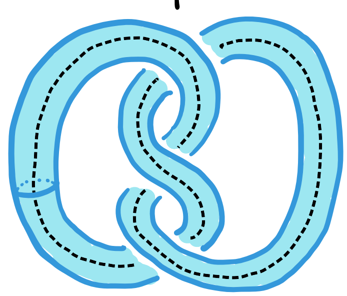
However, we usually don't use the Bennequin inequality to prove tightness... the standard route is via symplectic topology:

<u>Theorem (Eliashberg, Gromov).</u> If a contact structure can be filled by a compact symplectic manifold then it is tight.

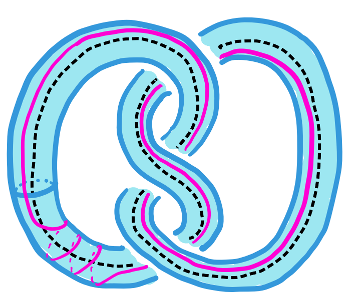
Contact Dehn surgery

Dehn surgery:

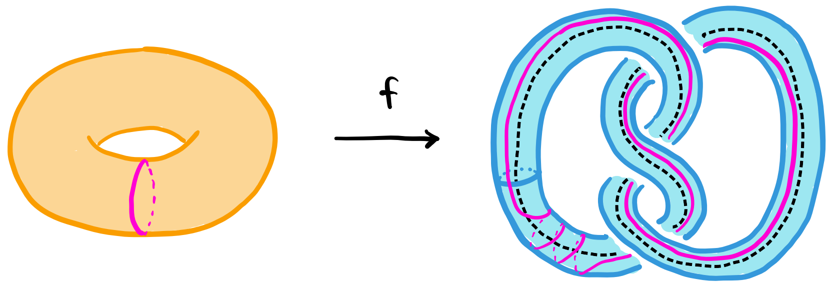
<u>Step 1.</u> Fix a knot γ in a 3-manifold M. It has a neighborhood N diffeomorphic to $S^1 \times D$.



<u>Step 2.</u> Fix an embedded curve α on $\partial N \subset \partial \overline{M \setminus N}$.



<u>Step 3.</u> Choose any diffeomorphism f of $T^2 = \partial(S^1 \times D^2)$ that sends the meridian $\{p\} \times \partial D^2$ to α.



<u>Last step.</u> Define the <u>α Dehn surgery along γ</u> to be the 3-manifold obtained from $\overline{M \setminus N}$ by gluing in a solid torus via f:

$$M(\gamma, \alpha) = (\overline{M \setminus N}) \cup_f (S^1 \times D).$$

Being careful, we can do the above with a tranverse knot in (S^3, ξ), where

$$\xi = \ker\left((x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)|_{S^3 \subset \mathbb{R}^4}\right),$$

such that the contact structure on $\overline{S^3 \setminus N}$ and a prescribed contact structure on the glued $S^1 \times D$ agree to give a contact structure on $M(\gamma, \alpha)$.

This construction extends to links, so this shows:

<u>Theorem (Martinet).</u> All closed compact 3-manifolds admit a contact structure.

Symplectic fillings

<u>Def.</u> A <u>symplectic form</u> ω on a manifold M is a closed, non-degenerate differential 2-form, i.e.

- $d\omega = 0$
- $\forall p \in M$: if $\exists X \in T_pM$ such that

$$\omega_p(X, Y) = 0$$

$\forall Y \in T_pX$, then $X = 0$.

[Equivalently, the skew-symmetric pairing on T_pM defined by ω is skew-symmetric. Skew-symmetric matrices are noninvertible in odd dimensions, so the existence of a symplectic form forces $\dim(M)$ to be even.]

A (positive) contact structure (M, ξ) is <u>symplectically fillable</u> if there exists a compact symplectic 4-manifold (X, ω) such that

(1) $\partial X = M$ and the boundary orientation on ∂X (induced from $\omega \wedge \omega$ on X) and the orientation on M agree, and

(2) $\omega|_\xi > 0$, i.e., if v_1, v_2 are an oriented basis for ξ at $p \in M$, then
$$\omega_p(v_1, v_2) > 0.$$

<u>Example</u>. The contact structure $(\mathbb{T}^3, \xi)$, where
$$\xi = \ker(\alpha = \sin(2\pi z)\,dx + \cos(2\pi z)\,dy),$$
is symplectically fillable. The 1-form
$$\beta_t = dz + t\alpha$$
is positive contact for $t > 0$:
$$\beta_t \wedge d\beta_t = (dz + t\alpha) \wedge t\,d\alpha = t^2 \alpha \wedge d\alpha > 0.$$
Let $\zeta_t = \ker \beta_t$. As $t \to \infty$, we see $\zeta_t \to \xi$. Hence ξ and ζ_t are isotopic. Notice that $\zeta_0 = \ker(dz)$ is the plane field parallel to the xy-plane.

Now consider $\mathbb{T}^3 = S^1 \times T^2$ as the boundary of $D^2 \times \mathbb{T}^2$ with the symplectic form $\omega = \omega_{D^2} + dx\,dy$, where ω_{D^2} is any area form of D^2.
$$\Rightarrow \quad \omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 1$$

$$\Leftrightarrow \quad \omega|_{\zeta_0} > 0$$
$$\Rightarrow \quad \omega|_{\zeta_t} > 0 \quad \text{for small } t$$
$$\Rightarrow \quad \omega|_{\xi} > 0.$$

<u>Corollary</u>. $(\mathbb{T}^3, \xi)$ is tight.

Another way to exhibit a symplectic filling is via Stein fillings:

Let X be a complex manifold with complex dimension 2. The complex structure on X induces a complex structure on TX, or equivalently, a bundle map $J: TX \to TX$ which is locally given by multiplication by i

From a function $\phi: X \to \mathbb{R}$, we can define a 2-form $\omega = d(d\phi \circ J)$ and a symmetric form
$$g(v,w) = \omega_p(v, Jw) \qquad (v,w \in T_pX)$$
If g is positive definite (i.e., is a Riemannian metric on X) the function ϕ is called <u>strictly plurisubharmonic</u>.

The manifold X is a Stein surface if X admits a proper such function. In this situation, the complex tangencies to $M_c = \phi^{-1}(c)$, i.e.

$$\xi_p = T_p M_c \cap J(T_p M_c), \quad (p \in M_c) \quad \left[\begin{smallmatrix}\text{plane}\\ \text{field}\end{smallmatrix}\right]$$

is a contact structure whenever c is a regular value. We call such a contact structure Stein fillable.