# Left orderability of Seifert fibered spaces.

$G$ grp.

**Def 1**: $G$ is LO if $\exists$ left invariant total order $<$ on $G$

$h<k \Leftrightarrow gh<gk \ \forall g \in G$.

- Ex: $G=\mathbb{Z}, \mathbb{Q}, \mathbb{R}$. non Ex: $\mathbb{Z}/n$
- LO $\Rightarrow$ torsion-free. $(1<g \Rightarrow g^n < g^n \cdot g = g^{n+1})$
- $G$ LO $\Rightarrow$ any $H$ LO $\forall H<G$. $\rightarrow \{-1,1\}$ $G\setminus\{id\}$

**Def 2**: $G$ is LO if $G = P \sqcup P^{-1} \sqcup \{1\}$ such that $P \cdot P \subseteq P$

Def $1 \rightarrow$ Def 2: $P = \{g \in G \mid g>1\}$. $(g>1,\ h>1 \Rightarrow gh>g>1)$

$\Downarrow$
$1>g^{-1}$

Def $2 \rightarrow$ Def 1: $g>h (\Leftrightarrow h^{-1}g>1) \Leftrightarrow h^{-1}g \in P$.

✱ • $1 \rightarrow N \rightarrow G \xrightarrow{\pi} Q \rightarrow 1$ if $N, Q$ LO then $G$ LO

Pf: $P_G = \pi^{-1}(P_Q) \sqcup P_N$.

(Cor: If $G = N \times Q$, $N, Q$ LO $\Rightarrow G$ LO.)

**Def**: $G$ is locally indicable (locally LO surjective) if $\forall_{1 \neq} H \overset{f.g}{<} G$

$H \twoheadrightarrow \mathbb{Z}$ ($H \twoheadrightarrow K$ for some $K \neq 1$ LO)

✱ Thm (Burns-Hale) LO $\Leftrightarrow$ locally LO surjective $\{1\}$ is LO.

$\Uparrow$

locally indicable

$\rightarrow$ $\Big\{$ $S$ finite symmetric $S\setminus\{1\} = P_S \sqcup N_S$.

$P_S \cap P_S^{-1} = \phi$ $\quad N_S \cap N_S^{-1} = \phi$

$P_S \cdot P_S \cap N_S = \phi$ $\quad N_S \cdot N_S \cap P_S = \phi$.

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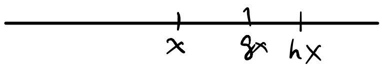
Chap 2

- **Ex**: $F_n$ loc. ind. $\Rightarrow$ LO.
- $* G_\lambda$ LO $\Leftrightarrow$ each $G_\lambda$ LO

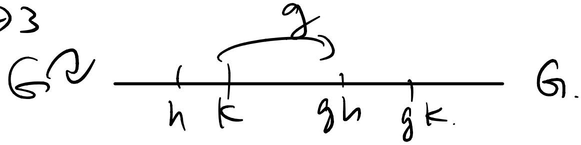
$\bigstar$ **POV3**: If $\exists\, \rho: G \to \mathrm{Homeo}^+(\mathbb{R})$ faithful then $G$ is LO.
Iff for $G$ countable

$3 \Rightarrow 1$



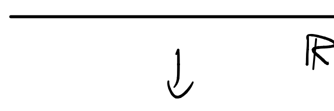

$g<h$ lexicographic
a countable dense set

No
$1 \Rightarrow 3$

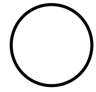



## $\curvearrowright S^1$ & $\curvearrowright \mathbb{R}$.

$\tilde{T} = \{\text{lifts of } f \in T\} \subseteq \mathrm{Homeo}^+(\mathbb{R})$




$T = \mathrm{Homeo}^+(S^1)$



$1 \to \mathbb{Z} \to \tilde{T} \to T \to 1$ central extension

$\rightsquigarrow eu \in H^2(T;\mathbb{Z})$

$\rho: G \to T$ $\quad eu_\rho = \rho^* eu \in H^2(G;\mathbb{Z})$

$eu_\rho = 0 \Leftrightarrow$ lifts to $\mathbb{R}$ action.

**Cor**: $\rho: G \to \mathrm{Homeo}^+(S^1)$, $H^2(G;\mathbb{Z}) = 0$ then $\tilde{\rho}: G \to \widetilde{\mathrm{Homeo}^+(S^1)} < \mathrm{Homeo}^+(\mathbb{R})$

$G = \pi_1(M^3)$.  $M$ cpt oriented.

- prime decomposition reduces to the case $M$ prime.

**Thm**: $M$ prime. Then $M \neq \mathbb{Q}HS \iff \pi_1 M$ Loc. ind. and $\neq 1$

$$\Downarrow$$

$\pi_1 M$ LO.

Q: when is $\pi_1 M$ LO if $M = \mathbb{Q}HS$? (L-sp. conj)

**Lemma**: $M$ irreducible, $1 \neq H < \pi_1 M$.
if (1) $[\pi_1 M ; H] = \infty$
or (2) $\partial M \neq \emptyset$
then $H \twoheadrightarrow \mathbb{Z}$

suppose $H_1(M; \mathbb{Q}) \neq 0$

$N \hookrightarrow M_H \rightarrow$ noncpt

$M_H \downarrow M$

$\partial N$

Lemma

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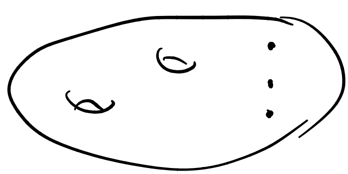
$M = SFS$ closed

$S^1 \rightarrow M \downarrow B$   almost $S^1$ bundle

exceptional

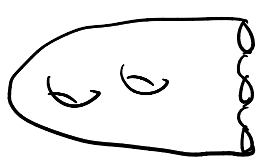
$S^1 \rightarrow M_0 \downarrow F$

F 

$N = M_0 \setminus \sqcup D_2 \times S^1 = M \setminus \sqcup D_2 \times S^1$

$S^1 \rightarrow N \downarrow F \setminus \sqcup D_2$

trivial



$$\pi_1 N = \pi_1(F \setminus \amalg D_\xi) \times \mathbb{Z}$$
$$= \left\langle a_1, b_1, \dots, a_g, b_g, \partial_1, \dots, \partial_b, z \;\middle|\; \begin{array}{l} [z, a_\xi] = [z, b_\xi] = [z, \partial_\xi] = 1 \\ [a_1, b_1] \cdots [a_g, b_g] \partial_1 \cdots \partial_b = z^k \end{array} \right\rangle$$

$$\pi_1 M = \pi_1 N \Big/ {z^{q_\xi} = \partial_\xi^{p_\xi}}$$

$\Rightarrow \langle z \rangle$ central and $1 \to \langle z \rangle \to \pi_1 M \to \pi_1 B \to 1$

B :


$\chi_0(B) = \chi(F \setminus \amalg D_\xi) + \sum \frac{1}{p_\xi}$

Fact: $\chi_0(B) < 0$
$\Rightarrow$ $\mathbb{H}^2$-struc on B
$\longrightarrow$ $\pi_1(B) \longrightarrow \mathrm{Isom}^+(\mathbb{H}^2)$ $F \setminus \amalg D_\xi$

$H_1(M; \mathbb{Q}) \cong \mathrm{span}_{\mathbb{Q}} \langle a_1, b_1, \dots, a_g, b_g \rangle$ QHS $\Rightarrow g = 0$
$\Rightarrow B = S^2(p_1, \dots, p_b)$

for B non-orientable

$$\pi_1 M = \left\langle a_1, \dots, a_g, \partial_1, \dots, \partial_b, z \;\middle|\; \begin{array}{l} a_\xi z a_\xi^{-1} = z^{-1}, \ [z, \partial_\xi] = 1 \\ a_1^2 \cdots a_g^2 \, \partial_1 \cdots \partial_b = z^k \\ z^{p_\xi} = \partial_\xi^{q_\xi} \end{array} \right\rangle$$

$H_1(M; \mathbb{Q}) \twoheadrightarrow H_1(B; \mathbb{Q}) \to \mathrm{rk} \geq g-1$ QHS $\Rightarrow g = 1$
$B = \mathbb{RP}^2(p_1, \dots, p_b)$

$\star$ $1 \to \langle z \rangle \to \pi_1 M \to \pi_1 B \to 1$ in both cases

$\langle z \rangle$ central iff B orientable

warning: $\langle z \rangle \not\cong \mathbb{Z}$

- M SFS QHS then $B = S^2(p_1, \dots, p_n)$ ①
$\mathbb{RP}^2(p_1, \dots, p_n)$ ②

**Thm 1:** In case ①, $\pi_1 M$ is LO iff $M = \mathbb{Z}HS$

**Thm 2:** In case ②, $\pi_1 M$ not LO

**general:** Boyer – Rolfsen – Wiest, characterize using foliation.

**for Thm1:** essential case: $\chi_o(B) < 0 \rightsquigarrow \rho: \pi_1 B \to \mathrm{Isom}^+(\mathbb{H}^2) \hookrightarrow \mathrm{Homeo}^+(S^1)$ faithful.

$$\begin{tikzcd}
1 \arrow[r] & \langle z \rangle \arrow[r] & \pi_1 M \arrow[r] \arrow[d, dashed] \arrow[dr, "\tilde{\rho}"] & \pi_1 B \arrow[r] \arrow[d, "\rho"] & 1 \\
& & \mathrm{Homeo}^+(\mathbb{R}) \arrow[r] & \mathrm{Homeo}^+(S^1) &
\end{tikzcd}$$

$$\underset{\substack{\cup \\ eu_{\tilde{\rho}}}}{H^2(\pi_1 M)} \overset{K(G,1)}{=} H^2(M) \overset{\mathbb{Z}HS}{=} 0$$

**For Thm2:**

$az = z^{-1}a \Rightarrow az^k = z^{-k}a.$

$$\pi_1 M = \left\langle a, d_1, \dots, d_b, z \;\middle|\; \begin{array}{l} aza^{-1} = z^{-1},\ [z, d_i] = 1 \\ a^2 d_1 \cdots d_b = z^k \\ z^{q_i} = d_i^{p_i} \end{array} \right\rangle$$

**Fact:** $a, z \neq 1$

WLOG $z > 1$

- If $a > 1$ then $a > z^k \quad \forall k \in \mathbb{Z}$.

  $\underline{z^{-k}a = az^k} > 1 \quad \forall k > 0 \Rightarrow a > z^k.$

  $a > 1 \geq z^k \quad \forall k \leq 0$

  for $b = 0$, $z^k = a^2 > a$ $\rightarrow\leftarrow$

  for $b > 0$, can make $p_i q_i > 0 \ \forall i \Rightarrow d_i > 1$

  $\Rightarrow a^2 d_1 \cdots d_b > 1.$