L-space conjecture

Thursday, October 7, 2021 9:23 PM

0. Abstract.
1. $\widehat{HF}$, Left-orderability (LO) & L-space conj
2. 1-bridge braids & (1,1)-knots
3. Technique
4. Result

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0. BGW: $\mathbb{Q}HS^3$ an L-space $\longleftrightarrow \pi_1$

1

1.1 $\widehat{HF}$

1.2. Def^n. A closed conn. ori. 3-mfld Y is an L-space, if

$$\operatorname{rk} \widehat{HF}(Y) = |H_1(Y, \mathbb{Z})|$$

1.3. Th^m. (Ozsváth, Szabó, 04)

$$\operatorname{rk} \widehat{HF}(Y) \geq |H_1(Y, \mathbb{Z})|$$

1.4. Def^n. A non-trivial gp G is LO if $\exists$ s.t.o, s.t.

$$g < h \Rightarrow fg < fh, \ \forall f, g, h \in G.$$

1.5. Conj. (Boyer, Gordon, Watson, 13).

An irr. $\mathbb{Q}HS^3$ is an L-spc. $\Leftrightarrow \pi_1$ is not LO.

($\Leftrightarrow$ doesn't admit taut foliation)

2.

2.1. Def^n. A knot $K \subseteq S^3$ is an L-space knot if $\exists$ non-trivial surgery on K yielding an L-space.

(i)$\Leftrightarrow$(ii)

L-space Conj. has been verified for some certain families

L-space Conj $(i) \Leftrightarrow (ii)$ has been verified for some certain families of knots, e.g.,

- Certain twisted torus knots
  (Christianson, Goluboff, Hamann, Varadaraj, 14)
- Certain 1-bridge braids
  (Liang, 17)
- Certain negatively twisted torus knots
  (Ichihara, Temma, 18)
- Any L-space obtained by surgeries on iterated 1-bridge braids
  (Nie, 20)
- Any L-space obtained by surgeries on (1,1)-knots
  (Nie, 21)

2.2 Def^n. A knot K in a closed, ori. 3-mfld Y is a (g,b)-knot, if $\exists$ a Heeg. splitting $Y = U_1 \cup U_2$ of genus g s.t each of $K \cap U_1$, $K \cap U_2$ consists b trivial arcs.
In particular, when $g=b=1$, the $(1,1)$ knots (sometimes 1-bridge torus knots) are widely studied.

2.3 Th^m. (Nie, 21) Let $Y = U_0 \cup_\Sigma U_1$ be a genus-1 Heeg. splitting of S^3 or $L(p,q)$. A knot $K \subseteq Y$ is a $(1,1)$ L-space knot, iff isotopic to a union of three arcs $\rho \cup \tau_0 \cup \tau_1$ s.t.

(i) ρ is a geod of Σ;
(ii) τ_0 is prop. $\hookrightarrow$ some meridinal disk in U_0;
(iii) τ_1 is . . . . U_1.

In particular, when τ_0 or τ_1 is of length 0, we have

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In particular, when T_0 or T_1 is of length 0, we have

2.4 Def^n (Greene, Lewallen, Vafaee, 16) ← a paper where this showed up, probably not the earliest time, studied in 90' by Berge & Gabai.

A knot K in the $D^2\times S^1$ is a 1-bridge braid, if isotopic to a union of two arcs $\rho\cup\tau$, s.t.

(i) $\rho\subseteq\partial(D^2\times S^1)$ is braided, i.e. $\pitchfork$ each meridian $\partial D^2\times pt$, &

(ii) τ is a bridge, i.e., prop $\hookrightarrow$ some meridional disk $D^2\times pt$.

It's positive if ρ is a positive braid in the usual sense.

A knot $K\subseteq S^3$ is a 1-bridge braid, if isotopic to a 1-bridge braid supported in a solid torus coming from a genus-1 Heeg. splitting of S^3.

2.5 Th^m (Nie, 21). Any non-trivial positive (1,1) L-space knot can be rep. by the closure of

$$(\sigma_w\sigma_{w-1}\cdots\sigma_{w-b_0+1})(\sigma_w\sigma_{w-1}\cdots\sigma_1)^{b_1}(\sigma_{w-1}\cdots\sigma_1)^{t-b_1}$$

in the braid gp B_{w+1} on $w+1$ strands, $1\le b_0\le w$, $1\le b_1\le t$.

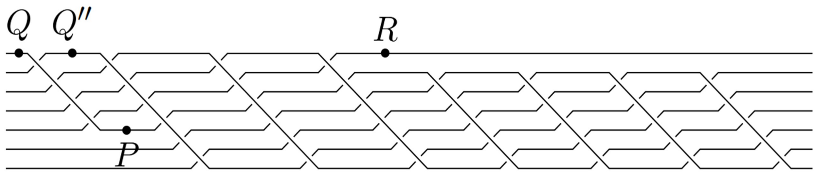

Figure 1: The braid when $(\omega, t, b_0, b_1) = (6, 7, 4, 3)$.

(Nie, 21)

2.6 Def^n. The braid gp w/ w strands is denoted by

$$\mathcal{B}_w=\langle\sigma_1,\sigma_2,\dots,\sigma_{w-1}\mid\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1},\ 1\le i\le w-2,\ \sigma_j\sigma_k=\sigma_k\sigma_j,\ 1\le j,k\le w-1,\ |j-k|\ge 2\rangle,$$

where σ_i gives strands i, $i+1$ a right-handed half-twist

2.7 Th^m (Classification Th^m of 1-bridge braids)

(Gabai, 90). Every 1-bridge braid is of the form

$$B(w,t+mw,b)=\mathcal{B}(w,t+mw,b),$$

i.e. the closure of

i.e. the closure of

$$\beta(w, t+mw, b) = (\sigma_1\sigma_2\cdots\sigma_b)(\sigma_1\sigma_2\cdots\sigma_{w-1})^{t+mw} \in \mathcal{B}_w.$$
$$1 \le b \le w-2,\quad 1 \le t \le w-2,\quad m \in \mathbb{Z}$$

We're interested when $B(w, t+mw, b)$ is a knot, instead a link!

It follows from Green, Lewallen & Vafaeé's more general result that

2.8. Th$\underline{m}$ (GLV, 16). All 1-bridge braids in S^3 are L-space knots.

In particular, positive 1-bridge braid admit positive L-space surgeries.

2.9. Examples. $B(w, t+mw, b)$ is a knot, if

(i) $t=1,\ b=2k,\ 1 \le k \le [(w-2)/2]$,

(ii) $w=2n+1,\ t=2n-1,\ b=2k,\ 1 \le k \le n-1$, or

(iii) $w=2n,\ t=2n-2,\ b=2k-1,\ 1 \le k \le n-1$.

3. The verification of the above examples relies on the following results.

3.1 Th$\underline{m}$ (Ozsváth, Szábo, 11). Let $K \subseteq S^3$ & suppose there exists $\frac{p}{q} \in \mathbb{Q}_{\ge 0}$ s.t. $S^3_{p/q}(K)$ is an L-space. Then, $S^3_{p/q'}(K)$ is an L-space iff $\frac{p'}{q'} \ge 2g(K)-1$.

Recall the following equiv. cond. from Joe's talk.

3.2. Th$\underline{m}$. Let G be a countable gp. Then the following are equiv. :

(i) G acts faithfully on the real line by order-pre. homeo.

(2) G acts faithfully on the real line by order-pre. homeo.

(ii) G is LO.

The following criterion generalizes Ichihara & Temma's result in 2015 proven w/ Th^{m} 3.2.

3.3. Th^{m} (Christianson, Goluboff, Hamann & Varadaraj, 16).

K a non-trivial knot $\subseteq S^3$. G the knot gp. $G(p/q)$ the quotient of G from p/q surgery. μ the meridian, s a v-frame long w/ $v>0$. Suppose G has two generators x, y s.t. $x=\mu$, s a word which excludes x^{-1}, y^{-1} & contains at least one x. Suppose further that $\forall$ homo $\Phi: G(p/q) \to Homeo^{+}(\mathbb{R})$ satisfies "$\Phi(x)t > t, \forall t \in \mathbb{R}$" $\Rightarrow$ "$\Phi(y)t \geq t, \forall t \in \mathbb{R}$". If $p, q > 0$, then, for $p/q \geq v$, $G(p/q)$ is not LO.

4.

4.1 Th^{m} $G(w, 1+mw, 2k, r)$, $G(2n+1, 1+m(2n+1), 2k, r)$ & non-LO, if $r \geq w+2k-1$, $r \geq 2n[2n-1+m(2n+1)]+2k$, & $r \geq (2n-1)(2n-2+2mn)+2k-1$, resp.