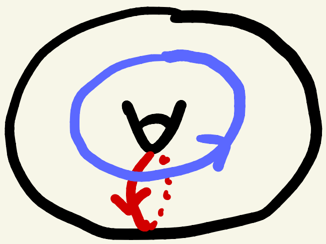
Seifert fibered spaces & L-spaces

9/24/2021

Outline

- basic notions of Dehn fillings (slopes, $\mathbb{Q}$-homology sphere/ solid torus...)
- triad (again)
- defining Seifert fibered spaces (SFS)
- SFS over base surface $\mathbb{RP}^2$ are L-spaces.

Dehn filling

Let T = 

* let μ and λ be oriented s.c.c.'s s.t. $\mu \cdot \lambda = -\lambda \cdot \mu = 1$

$[\mu]$ and $[\lambda]$ form a basis for $H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$.

• α, β unoriented s.c.c's on T;

α and β are isotopic $\iff$ $[\alpha] = \pm[\beta] \in H_1(T)$

A slope on T is an isotopy class of unoriented essential s.c.c.'s on T.

- α is a slope on T

$\Rightarrow$ $\alpha = \pm(p[\mu] + q[\lambda])$ s.t.

p, q are coprime.

$$\{\text{slopes on } T\} \overset{1:1}{\longleftrightarrow} \{\mathbb{Q} \cup \{\tfrac{1}{0}\}\}$$

The distance $\Delta(\alpha, \beta)$ between two slopes is their minimal geometric intersection number.

Let $\alpha = \frac{p}{q}$ and $\beta = \frac{r}{s}$,

$$\Delta(\alpha, \beta) = \Delta\left(\frac{p}{q}, \frac{r}{s}\right) = |ps - qr|.$$

Let M be a 3-mfd w/ $\partial M \cong T$,
α be a slope on ∂M
(for some fixed basis for $H_1(\partial M)$)

Define α-Dehn filling on M, $M(\alpha)$

$$M(\alpha) = M \cup_{\varphi} V$$

where V is a solid torus,
($\cong S^1 \times \mathbb{D}^2$)

and
$$\varphi : \partial V \longrightarrow \partial M$$
$$\partial(\{pt\} \times \mathbb{D}^2) \longmapsto \alpha$$

Remarks:

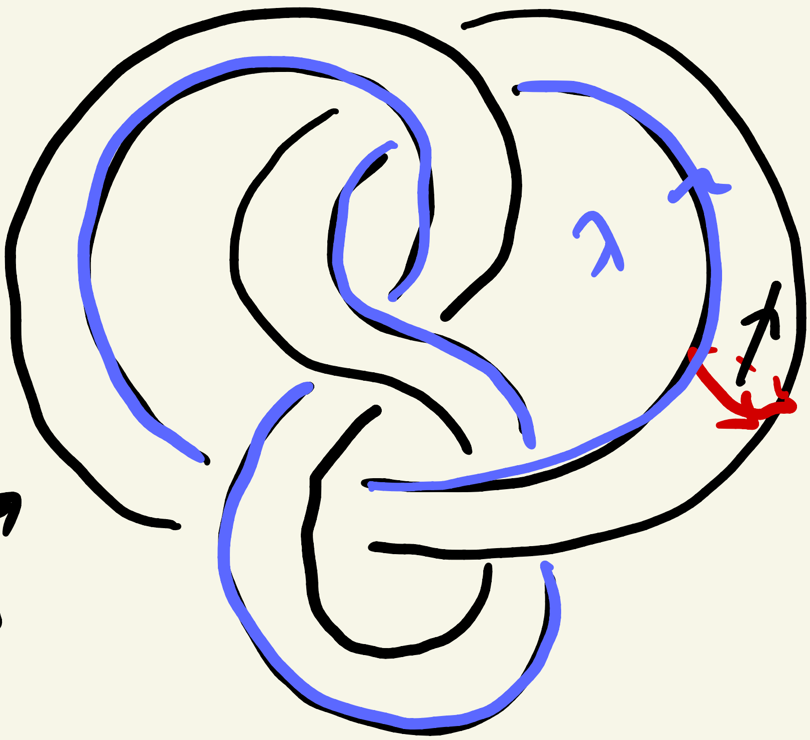
• To say $\alpha = \frac{p}{q} \in \mathbb{Q} \cup \{\frac{1}{0}\}$,

you need to specify / fix a basis for $H_1(\partial M)$.

e.g. knot exteriors (in S^3) $X_K := S^3 \setminus (\nu(K))$

regular nbhd of K

$\nu(K)$



=

$w = \pm 3$

$$S^3_{p/q}(K) = X_K(p/q)$$

$H_1(\partial X_K)$ is gen. by $[\mu_K]$ & $[\lambda]$

Seifert framing i.e. bounds a Seifert surface.

Let M be a $\mathbb{Q}$-homology solid torus,
i.e. $\partial M \cong T$ & $H_*(M;\mathbb{Q}) \cong H_*(S^1 \times \mathbb{D}^2; \mathbb{Q})$.

(i.e. $H_1(M) \cong \mathbb{Z} \oplus H$)

finite abelian group

let $i_*: H_1(\partial M; \mathbb{Q}) \to H_1(M; \mathbb{Q})$ be the homomorphism incuced by the inclusion map $i: \partial M \hookrightarrow M$.

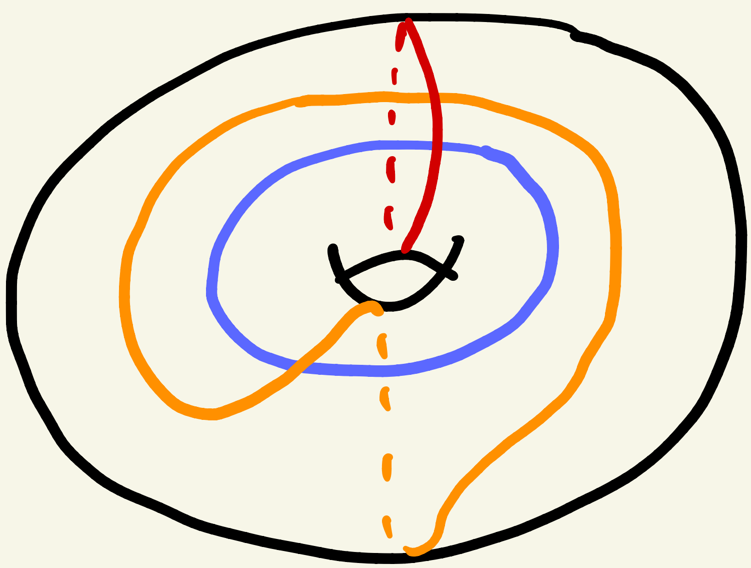
Then there exists a unique slope on ∂M called the rational longitude λ_M s.t. $i_*([\lambda_M])$ is zero.

<u>Remark</u>: λ_M is the unique slope s.t.

$H_2(M(\lambda_M)) \neq 0$.

Triad

Let M be a 3-mfd with $\partial M \cong T$, & α, β and γ be slopes on ∂M s.t. $\Delta(\alpha,\beta) = \Delta(\beta,\gamma) = \Delta(\gamma,\alpha) = 1$.



Then (α, β, γ) are said to be a triad of slopes.

A $(M(\alpha), M(\beta), M(\gamma))$ forms **a triad of 3-manifolds**

For some cyclic order,

$$|H_1(M(\gamma))| = |H_1(M(\alpha))| + |H_1(M(\beta))| \quad *_1$$

e.g. For α, β s.t. $\Delta(\alpha, \beta)$, $\alpha + \beta$.
$(\alpha, \beta, \alpha+\beta)$ in a triad.

$\Rightarrow$ they fit in a surgery exact triangle $*_2$

$$\begin{tikzcd}
 & \widehat{HF}(M(\beta)) \arrow[d] \\
\widehat{HF}(M(\alpha)) \arrow[ur] & \widehat{HF}(M(\gamma)) \arrow[l]
\end{tikzcd}$$

By rank & nullity and exactness in $*_1$, $*_2$ we immediately obtain the following.

Proposition 1: If M admits a triad $(\alpha, \beta, \alpha+\beta)$ s.t. $M(\alpha)$ and $M(\beta)$ are L-spaces, $M(\alpha+\beta)$ is an L-space as well.

Inducting on the length of a continued fraction of $p/q \in \mathbb{Q}$ with $p, q \geq 0$ and p and q coprime, we get

Proposition 2: ——— (as above).

Then for all coprime pairs $p, q \geq 0$, $M(p\alpha + q\beta)$ is an L-space.

Corollary: If $S^3_r(K)$ is an L-space for some $r \in \mathbb{Q}_{\geq 0}$, then $S^3_s(K)$ is an L-space as well for all $s \geq r$.

Seifert fibered spaces

We'll define the Seifert manifolds as Dehn fillings of trivial bundles over surfaces with boundary.

Let N be the unique oriented bundle $F_0 \tilde{\times} S^1$, (possibly nonoriented)
& $T_1, \dots, T_n$ be the components of ∂N.

$\Rightarrow$ fix a basis s_i, f_i for $H_1(T_i)$

$s_i \to F_0 \times \{pt\}$ (∂F_0); $f_i \to \{pt\} \times S^1$

Definition: A Seifert manifold is any 3-mfd M

$$M = N\left(\frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}\right)$$ with $k \leq n$

and $p_i \neq 0$.

Let F be F_0 with k boundary comp.s capped.
(base surface)

then we denote M as

$$\boxed{M := F\left(\frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}\right).}$$

Remarks:

- may orient f_i s.t. $p_i \geq 1$
- $\mathbb{D}^2(p/q)$ is a solid torus.
- $S^2(p/q)$ is the lens space $L(q,p)$
  (S^3 as a lens sp)
- $S^2(p_1/q_1, p_2/q_2)$ is also a lens space
- $\frac{p_i}{q_i} = \frac{1}{0}$ ← just ignore this.

twisting along an annulus

- $k \geq 2$, $\left(\frac{p_i}{q_i}, \frac{p_j}{q_j}\right) \longmapsto \left(\frac{p_i}{q_i + p_i}, \frac{p_j}{q_j - p_j}\right)$

$\therefore$ $\frac{1}{n}$ → ignore that too(!)

- M is a $\mathbb{Q}HS^3$ if $F = S^2$ or $\mathbb{RP}^2$.

$[\pi_1(M) \twoheadrightarrow \pi_1(F)]$

[Spoiler: The case $F = S^2$ will be done by Allie on November 12!]

$F = \mathbb{RP}^2$

$M = \mathbb{RP}^2(a_1, \ldots, a_n)$

we'll induct on n.

Base case $n=1$:

Möbius band

$$M = \mathbb{RP}^2(a) = (M \tilde{\times} S^1)(a)$$

(see last two pages)

$$= \mathbb{D}^2(2/1, 2/1)(a)$$

$$= S^2(\overset{+}{\tfrac{2}{1}}, \overset{-}{\tfrac{2}{1}}, a)_{\hookrightarrow p/q}$$ "prism mfd"

$S^2\left(\frac{2}{1-2}, \frac{2}{1+2}, p/q\right)$ elliptic mfds

$\overset{\text{s}\shortparallel}{}$ $S^2\left(-2, \frac{2}{3}, p/q\right)$ (spheical)

$\hookrightarrow \pi_1$ finite

<u>Thm (Oz-Sz)</u>: Elliptic mfds are L-spaces

<u>Induction hypothesis:</u>

(*) $\mathbb{R}P^2(a_1, \dots, a_r)$ $r < n$

$\hookrightarrow$ is an L-space.

<u>note:</u>

— $\mathbb{R}P^2(\underbrace{a_1, \dots a_r, \tfrac{1}{k}}_{r+1}) = \underbrace{\mathbb{R}P^2(a_1', \dots, a_r')}_{r}$

$M = \mathbb{R}P^2(a_1, \dots, a_n')$.

let N be the 3-mfd obtained by

$N = \mathcal{U}(a_1, \dots, a_{n-1})$.

$\Rightarrow \quad M = N(a_{n-1})$

$* \quad N(s_n) = \mathbb{RP}^2(a_1, \dots, a_{n-1})$

section

is an L-space by (*)

$*$ N is a $\mathbb{Q}$-homology solid torus.

$\Rightarrow H_2(N(\lambda_N)) \neq 0$ \
but $H_2(N(f_n)) \neq 0$

$\Big\}$ $\lambda_N = f_n$.

$N(s_n + f_n) = N(1) = \mathbb{RP}^2(a_1, \dots, a_{n-1}, 1)$

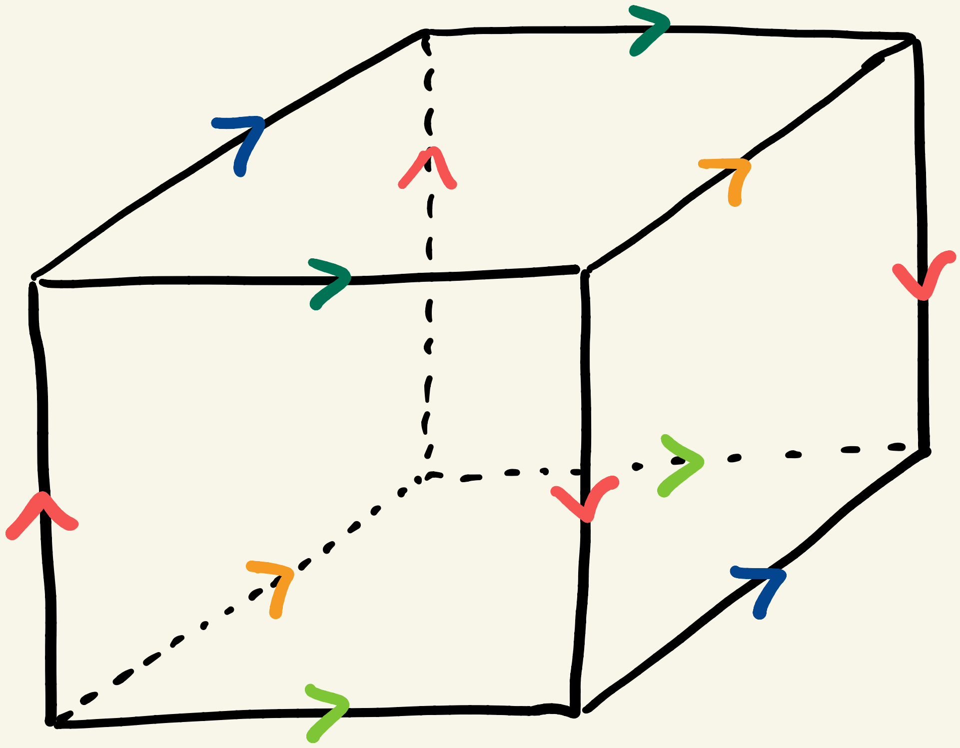
$= \mathbb{RP}^2(a_1', \dots, a_{n-1}')$

is an L-space by (*)

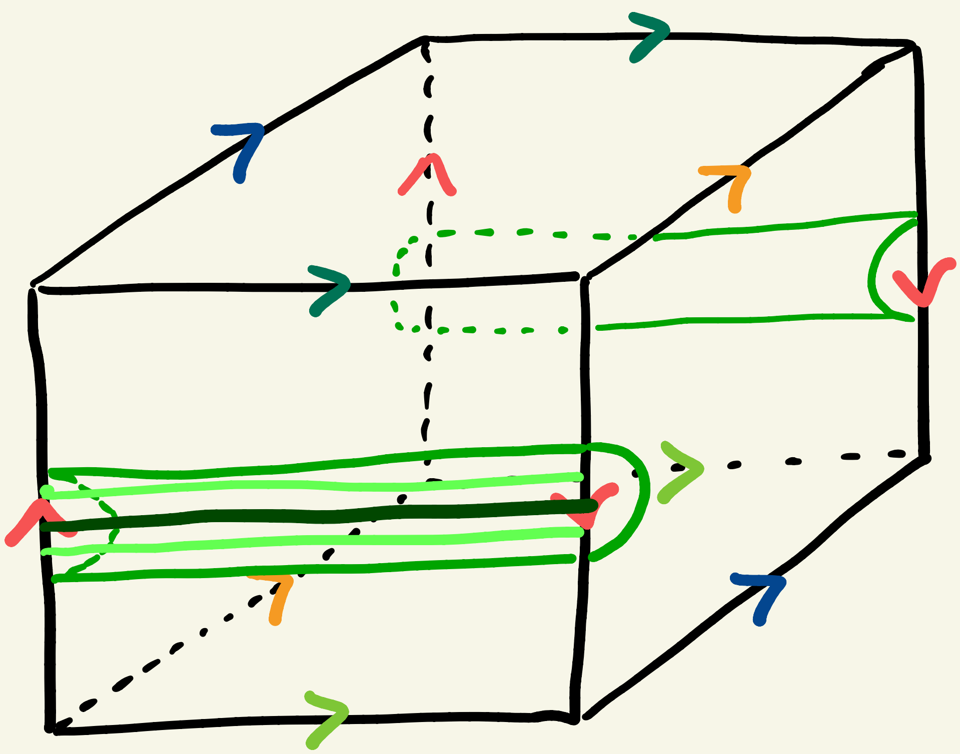
By Proposition 2,

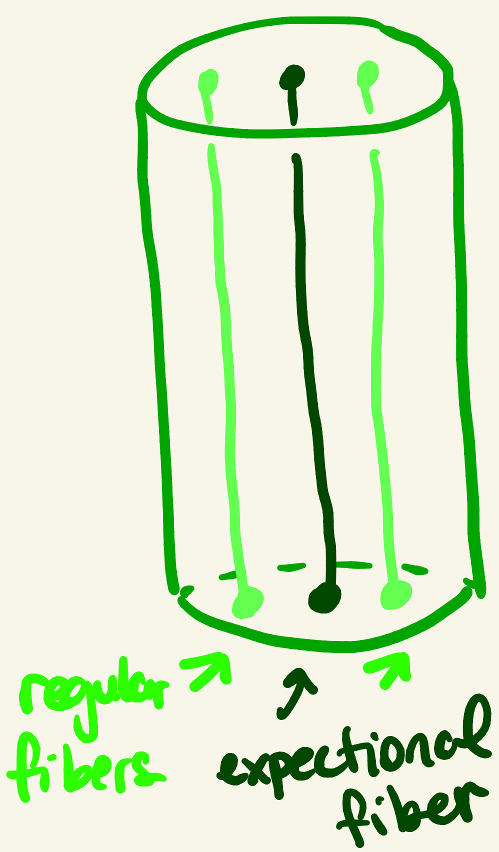
$N(p(s_n) + q(s_n + f_n))$ is an L-space. $\square$

Bonus: $M \tilde{\times} S^1 \cong \mathbb{D}^2\left(\frac{2}{1}, \frac{2}{1}\right)$

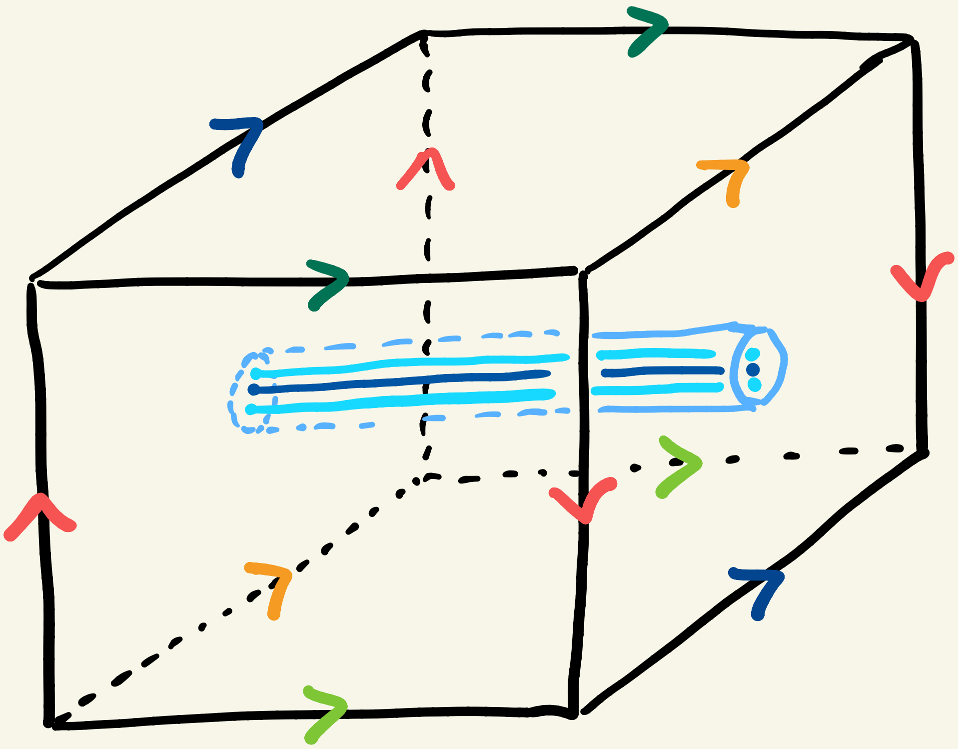


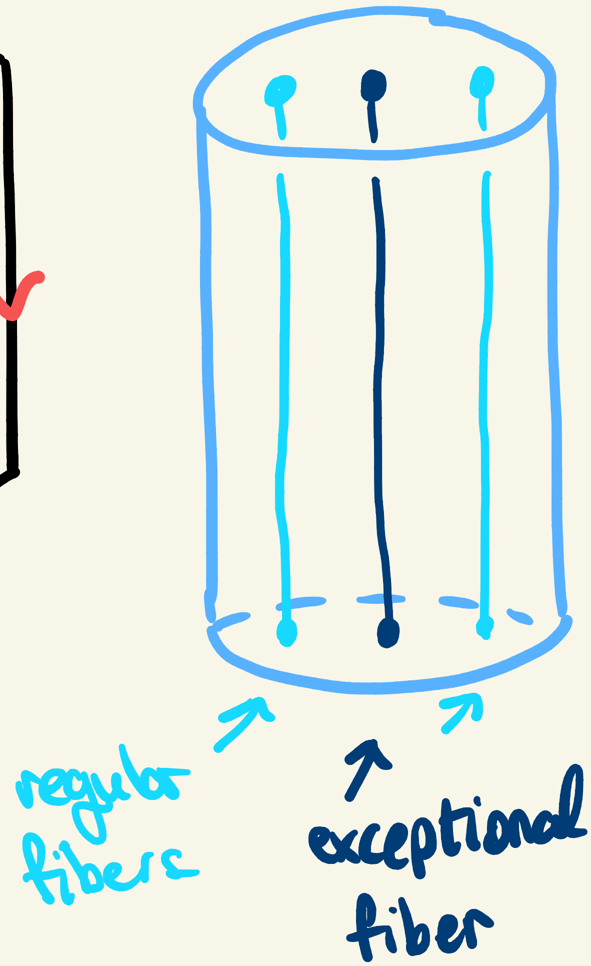
fibered solid torus nhood 1:

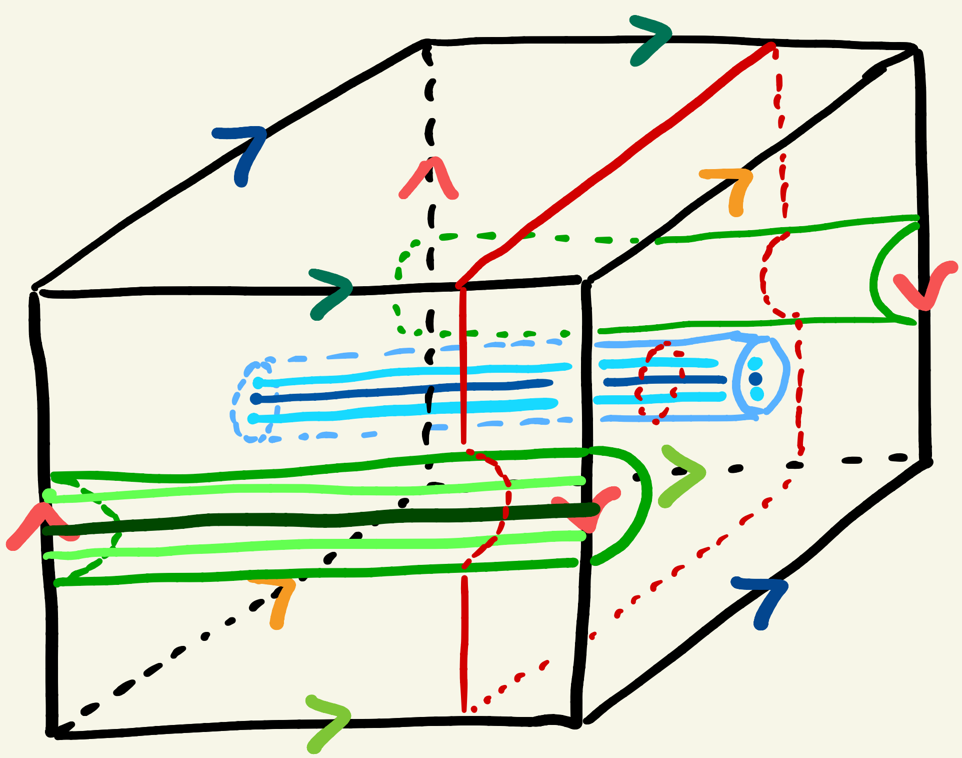


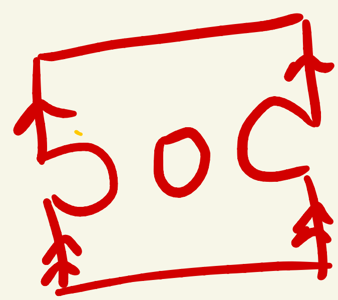


fibered solid torus nbhd 2:





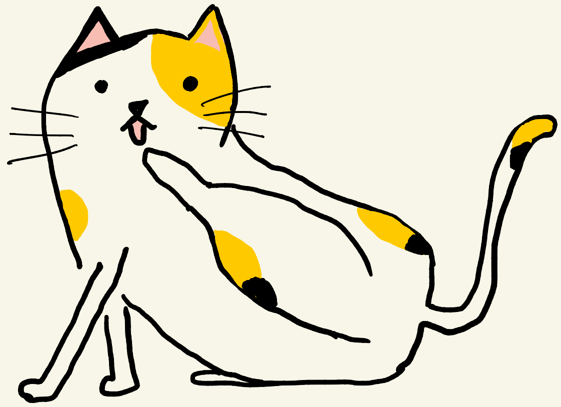


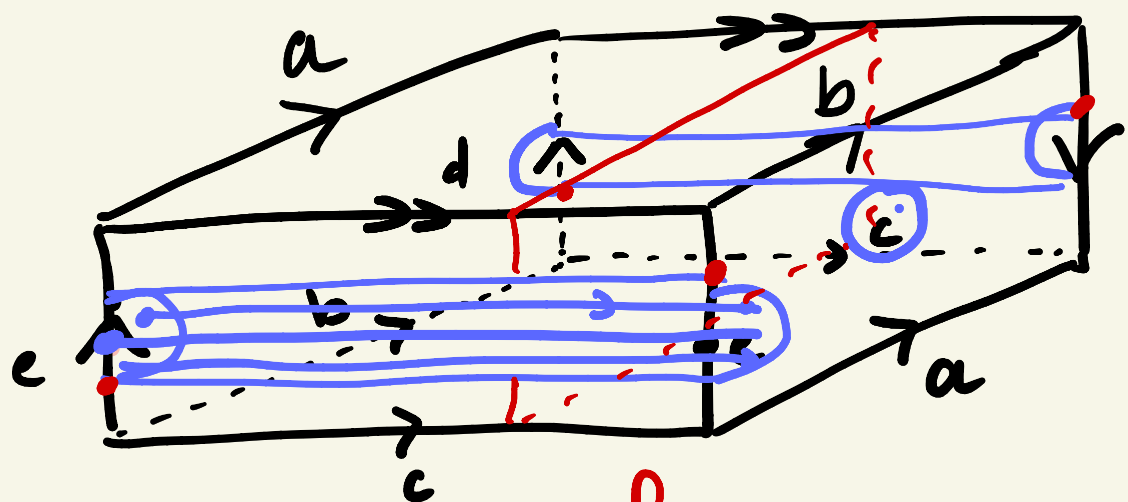


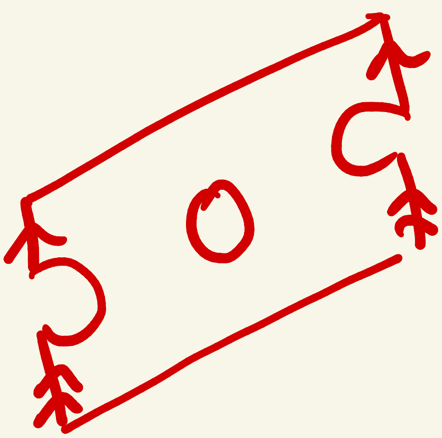
pair of pants!

(not a slice of swiss cheese)

Thank you!







$\simeq$ pair of pants!