Intro to Heegaard Floer Homology and L-spaces

9/10/2021

Outline:

- What is HF & what does it tell us?
- 2 definitions for L-spaces
- How do we compute HF?
- Examples
  (show S^3 & lens spaces are L-spaces)

What is HF & what does it tell us?

History & Purpose:

- $\{\widehat{HF}, HF^+, HF^-, HF^\infty, HF_{red}\}$ a package of 3-manifold invariants defined by Ozvath & Szabo in early 2000's
- HFK defined by Ozvath & Szabo and independently Rasmussen
  - → categorifies Alexander polynomial
  - → detects Seifert genus & fiberedness

What kind of objects do we use?

Conventions & notation

Know: All 3-mflds, Y, have a Heegaard decomposition, $\mathcal{H}$. We let $Y = \mathbb{Q}HS^3$ & irreducible

→ computing HF relies on $\mathcal{H}$ but is independent of choice of splitting ($\mathcal{H}$).

We let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ throughout (for simplicity)

We let $\mathcal{U}$ be a formal variable (of degree -2)

goal: define "flavors" $\widehat{HF}$, HF^-, HF^+, HF^∞, HF_{red}

(HF° in general)

Fact: $HF^\circ(Y) = \bigoplus_{s \in spin^c(Y)} HF^\circ(Y, s)$

Define
$spin^c(Y)$ = homotopy classes of non-vanishing vector spaces on Y (if Y has boundary) or $Y - B^3$ (if Y is closed)

Fact: $spin^c(Y) \xleftrightarrow[\text{bijection}]{} H^2(Y; \mathbb{Z}) \cong H_1(Y; \mathbb{Z})$

for Y closed, oriented only

Chain complexes:

$\widehat{CF}(\mathcal{H})$ over $\mathbb{F}$ $\qquad$ $CF^-(\mathcal{H})$ over $\mathbb{F}[U]$

$\widehat{HF}$, HF^- are homologies of $\widehat{CF}$, CF^-

$\widehat{HF}$ is finitely generated & graded vector space over $\mathbb{F}$

$$\widehat{HF}(Y,s) \cong \bigoplus_i \mathbb{F}_{(d_i)}$$

HF^- is a finitely generated & graded module over $\mathbb{F}[U]$

$$HF^-(Y,s) \cong \underbrace{\mathbb{F}[U]_{(d)}}_{} \oplus \bigoplus_j \mathbb{F}[U]_{(c_j)} / (U^{n_j})$$

since $Y \cong QHS^3$ there is only one term here

<u>d-invariant</u>, $d(Y,s) = \max\{gr(x) \mid x \in HF^-(Y,s)$ & $U^N x \neq 0 \ \forall N > 0\}$

$= d$

Also,

$$HF_{red}(Y,s) = \bigoplus_j \mathbb{F}[U]_{(c_j)} / (U^{n_j})$$

- torsion part of HF^-

L-space def 1 Y is an L-space iff

$HF_{red}(Y,s) = 0 \quad \forall s \in \text{spin}^c(Y)$

Finally,

$$CF^\infty(Y,s) = CF^-(Y,s) \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$$

$CF^- \leq CF^\infty$, CF^- generated by elements of grading ≤ -2

$\therefore$ $CF^+ = CF^\infty / CF^-$

Two definitions of L-spaces

How are HF^-, $\widehat{HF}$ related?

we have short exact sequence:

$$0 \to CF^-(\mathcal{H},s) \xrightarrow{U} CF^-(\mathcal{H},s) \longrightarrow \widehat{CF}(\mathcal{H},s) \longrightarrow 0$$

$\Rightarrow$ exact triangle:

$$\begin{tikzcd} HF^-(Y,s) \arrow[rr, "U"] & & HF^-(Y,s) \arrow[dl] \\ & \widehat{HF}(Y,s) \arrow[ul] & \end{tikzcd}$$

Lemma: $\dim \widehat{HF}(Y;s) \geq |H_1(Y;\mathbb{Z})|$

Proof: Say $\dim \widehat{HF}(Y) < |H_1(Y,s)|$

$$\dim \widehat{HF}(Y) = \sum_{\mathrm{spin}^c(Y)} \dim \widehat{HF}(Y,s)$$

Since $|\mathrm{spin}^c(Y)| = |H_1(Y,\mathbb{Z})|$, this tells us $\widehat{HF}(Y,\mathfrak{s}) = 0$ for some $\mathfrak{s} \in \mathrm{spin}^c(Y)$

Then, substitute into exact triangle

$$HF^-(Y,\mathfrak{s}) = \mathbb{F}[U]_{(d)} \oplus \bigoplus_j \mathbb{F}[U]_{(c_j)}/(U^{n_j})$$

$$\begin{tikzcd}
\mathbb{F}[U]_{(d)} \oplus \bigoplus_j \mathbb{F}[U]_{(c_j)}/(U^{n_j}) \arrow[rr, "U"] & & \mathbb{F}[U]_{(d)} \oplus \bigoplus_j \mathbb{F}[U]_{(c_j)}/(U^{n_j}) \arrow[dl, "\alpha"] \\
& 0 \arrow[ul, "\beta"] &
\end{tikzcd}$$

$\Rightarrow U$ is an isomorphism, which isn't true since $\mathrm{im}\, U$ does not contain any constants.

$\square$

Def 2: Y is an L-space if $\dim \widehat{HF} = |H_1(Y;\mathbb{Z})|$

Def 1: Y is an L-space if $HF_{red}(Y,s)=0\ \forall s$

Corollary: Def 1 $\Leftrightarrow$ Def 2

Proof:

Say $HF_{red}(Y,s)=0\ \forall s \in spin^c(Y)$

Then $HF^-(Y,s) = \mathbb{F}[u]_{(d)}\ \forall s$

Have:

$$\begin{tikzcd} \mathbb{F}[u] \arrow[rr, "u"] & & \mathbb{F}[u] \arrow[dl, "\alpha"] \\ & \widehat{HF}(Y,s) \arrow[ul, "\beta"] & \end{tikzcd}$$

$\cong \mathbb{F}^k,\ k>0$

$\ker u = 0$

$\operatorname{im} u = \deg_u \geq 1$ polys

$\ker \alpha = \deg_u \geq 1$ polys $(u \mapsto 0)$

$\operatorname{im} \alpha =$ constants

$\ker\beta$ = constants

$\operatorname{im}\beta = 0$ $\Rightarrow K = 1.$

$(= \ker U)$

Thus, $\widehat{HF}(Y,s) = \mathbb{F}$ & $\dim \widehat{HF}(Y) = |H_1(Y,s)|$

Now, say $\dim \widehat{HF}(Y) = |H_1(Y,s)|$

For a contradiction, say $HF_{red}(Y,s) \neq 0$ for some $s \in spin^c(Y)$.

Example: let $HF^-(Y,s) = \mathbb{F}[U] \oplus \mathbb{F}[U]/(U^2)$

Then our exact triangle gives

$$\begin{tikzcd} \mathbb{F}[U] \oplus \mathbb{F}[U]/(U^2) \arrow[rr, "U"] & & \mathbb{F}[U] \oplus \mathbb{F}[U]/(U^2) \arrow[dl, "\alpha"] \\ & \mathbb{F} \arrow[ul, "\beta"] & \end{tikzcd}$$

$\ker U = (0, \quad U \quad \cong \mathbb{F} \quad) = \operatorname{img} \beta$

$\operatorname{img} U = (\underset{\text{w/o const}}{\mathbb{F}[U]}, \quad U \quad \cong \mathbb{F} \quad) = \ker \alpha$

$\operatorname{img} \alpha = (\underset{\cong \mathbb{F}}{\text{const only}}, \quad 1, U+1 \quad) = \ker \beta$

$$1 = \dim \mathbb{F} = \underbrace{\dim \operatorname{img} \beta}_{1} + \underbrace{\dim \ker \beta}_{>1} > 1 \quad \#$$

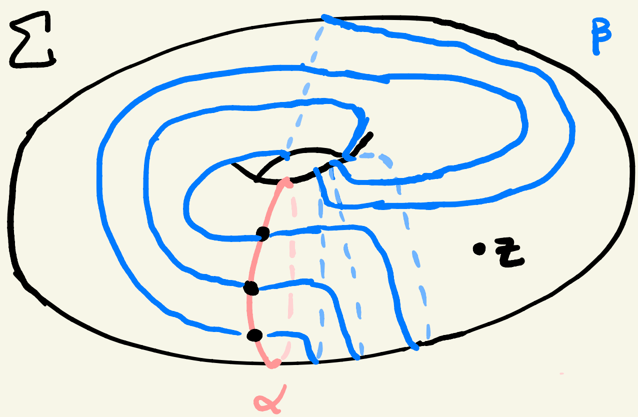
so $HF_{red}(Y,s) = 0 \quad \forall\, s \in spin^c(Y)$ $\square$

How do we compute HF?

Let $\mathcal{H} = (\Sigma, \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, z)$ be a <u>pointed</u> Heegaard diagram

basepoint $z \in \Sigma - \vec{\alpha} - \vec{\beta}$

<u>Example $\mathcal{H}$</u>



<u>In general</u>

Define $Sym^g(\Sigma) = \Sigma^g / S_g =$

$\{$unordered g-tuples of points in $\Sigma\}$

(If $g=1$, $Sym^1(\Sigma) = \Sigma$)

$Sym^g \Sigma$ is a smooth $2g$-manifold

Define $V_z = \{z\} \times Sym^{g-1}\Sigma \subseteq Sym^g \Sigma$

= {unordered g-tuples on Σ where at least one point is z}

we look at submflds:

$$\mathbb{T}_\alpha = \prod_{i=1}^{g} \alpha_i \quad , \quad \mathbb{T}_\beta = \prod_{i=1}^{g} \beta_i$$

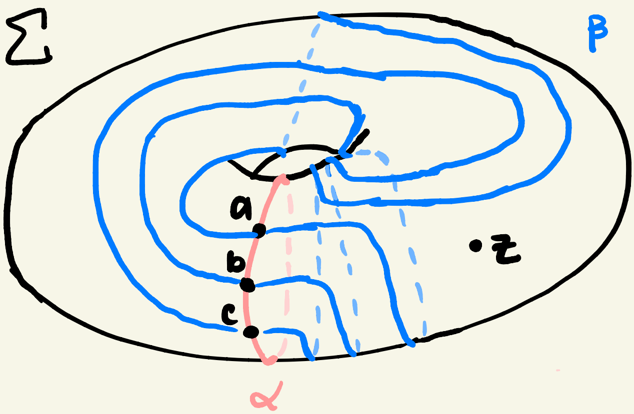
Both are g-dimensional tori in $Sym^g \Sigma$

(If $g=1$ $\mathbb{T}_\alpha = \alpha$, $\mathbb{T}_\beta = \beta$)

$\mathbb{T}_\alpha \cap \mathbb{T}_\beta = \{\{x_1, \ldots, x_g\} \mid x_i \in \alpha_i \cap \beta_{\sigma(i)}\}$

generates our chain complex.

Examples



$\widehat{CF}(\mathcal{H}) = \langle a, b, c \rangle$

$= \mathbb{F}^3$

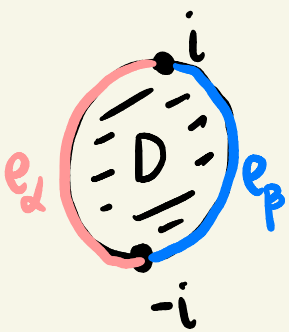
$CF^-(\mathcal{H}) = \langle a, b, c \rangle$

$= \mathbb{F}[U] \oplus \mathbb{F}[U] \oplus \mathbb{F}[U]$

What is the differential?

A Whitney disk from x to y is a map

$\phi : \mathbb{D} \to \mathrm{Sym}^g \Sigma$ such that:

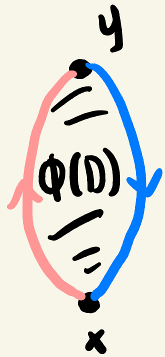


$\phi(i) = y$

$\phi(e_\beta) \subseteq \mathbb{T}_\beta$

$\phi(e_\alpha) \subseteq \mathbb{T}_\alpha$

$\phi(-i) = x$



let $\pi_2(x,y)$ be the set of homotopy classes of whitney disks from x to y.

let $n_z(\phi) = \#(\phi(\mathbb{D}) \cap V_z)$

(intersection number of whitney disk with basepoint z)

let $\mathcal{M}(\phi)$ = moduli space of (pseudo-) holomorphic representatives of ϕ

& Maslov index $\mu(\phi)$ = expected dimension of $\mathcal{M}(\phi)$

when $\mu(\phi)=1$, $\mathcal{M}(\phi)/\mathbb{R} = \hat{\mathcal{M}}(\phi)$

$\mathcal{M}(\phi)$ mod ($\mathbb{R}$-action of $\mathbb{C}$-automorphisms fixing $\pm i$) is a 0-dimensional compact manifold. $\#\hat{\mathcal{M}}(\phi)$ = number of points.

Now, $\hat{\partial} : \widehat{CF}(\mathcal{H}, s) \to \widehat{CF}(\mathcal{H}, s)$

$$\hat{\partial} x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} \# \hat{M}(\phi) \cdot y$$

And, $\partial^- : CF^-(\mathcal{H}, s) \to CF^-(\mathcal{H}, s)$

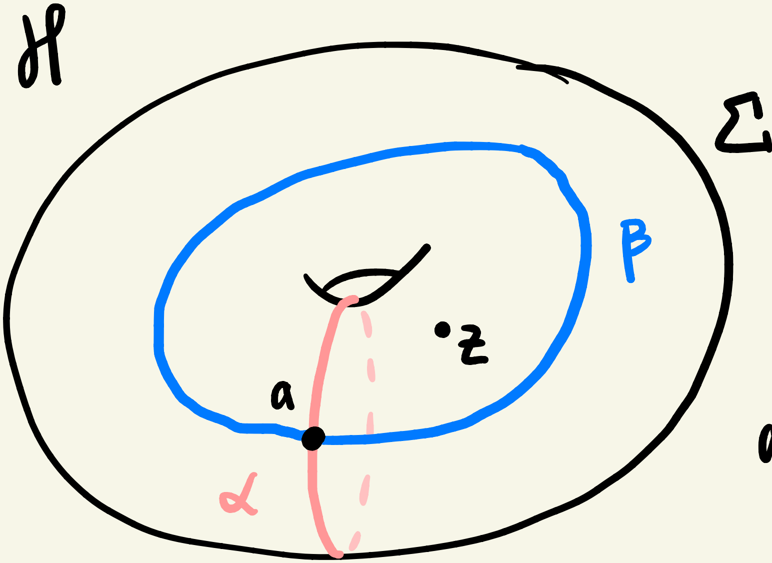
$$\partial^- x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \# \hat{M}(\phi)\, U^{n_z(\phi)} \cdot y$$

can get $\widehat{CF}$ from CF^- setting $U=0$

<u>Theorem</u> $\hat{\partial}^2 = \partial^{-2} = 0$ & HF° is an invariant of Y.

Examples

$\boxed{S^3}$



$a \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$

$\widehat{CF}(Y) = \mathbb{F}$

$CF^-(Y) = \mathbb{F}[\mathcal{U}]$

only whitney disk is trivial & $\mu(\phi) = 0$ for ϕ trivial

alt: ∂ lowers grading by 1, so $\partial a = 0$. $\Rightarrow$ $\hat{\partial} a = 0$.

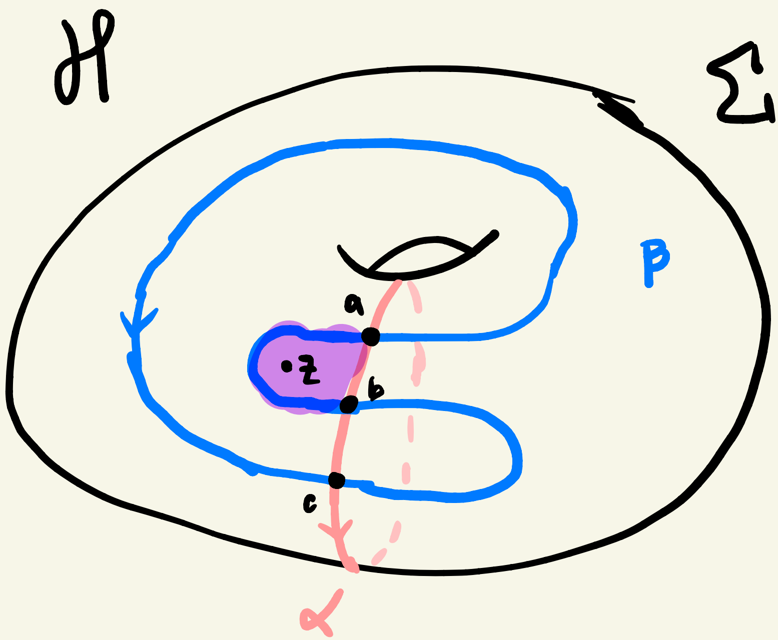
$$\hat{\partial} x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} \# \hat{M}(\phi) \cdot y$$

$\widehat{HF}(Y) = \mathbb{F}$

$HF^-(Y) = \mathbb{F}[\mathcal{U}]$

$\rightsquigarrow S^3$ is an L-space.

$\boxed{S^3}$ Note: $H_1(S^3)$ is trivial, so $|\text{spin}^c(Y)| = 1$.



$\mathbb{T}_\alpha \cap \mathbb{T}_\beta = \{a, b, c\} \rightsquigarrow \widehat{CF}(Y) = \mathbb{F}^3$

$CF^-(Y) = \mathbb{F}[U] \oplus \mathbb{F}[U] \oplus \mathbb{F}[U]$

$$\hat{\partial} x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} \#\widehat{M}(\phi) \cdot y$$

$\hat{\partial} a = 0$

$\hat{\partial} c = b$

$\Rightarrow \hat{\partial} b = 0$

$\operatorname{Ker} \hat{\partial} = \langle a, b \rangle$

$\operatorname{img} \hat{\partial} = \langle b \rangle$

$\Rightarrow \widehat{HF}(Y) = \mathbb{F}$

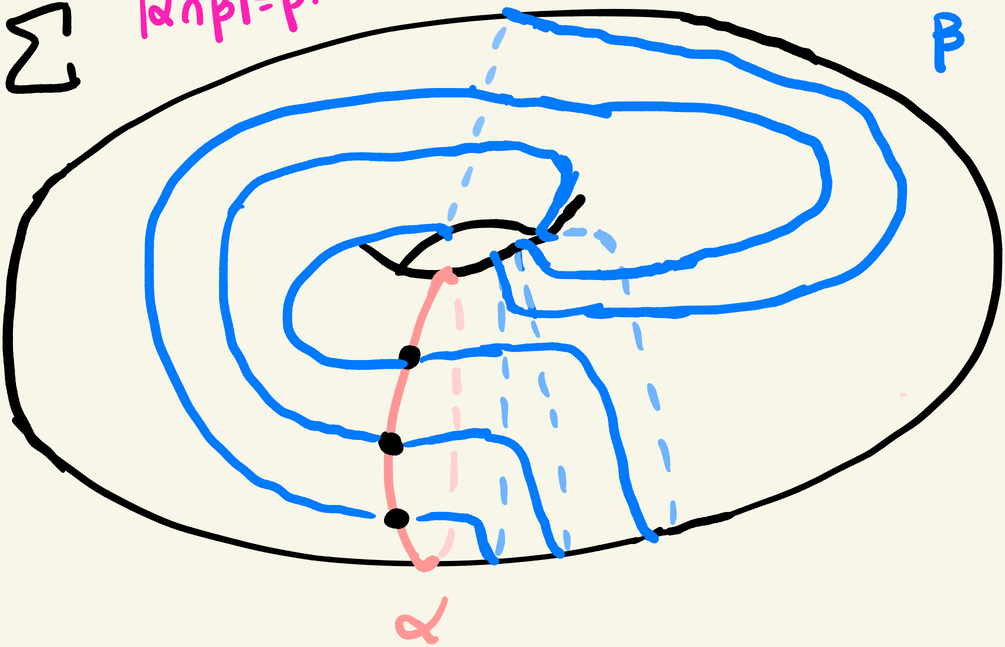
$\boxed{Y = L(3,2)}$

$H_1(Y;\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$

so $|spin^c(Y)| = 3$

can always get a diagram where $|\alpha \cap \beta| = p$.

Σ



$|\mathbb{T}_\alpha \cap \mathbb{T}_\beta| = |spin^c(Y)|$

Thus, $\widehat{CF}(Y, s_i) = \langle x_i \rangle$

$\Rightarrow \hat{\partial}, \partial^- = 0$ (trivial whitney disks)

$\Rightarrow \widehat{HF}(Y, s_i) = \mathbb{F} \Rightarrow \widehat{HF}(Y) = \mathbb{F}^{|spin^c Y|}$

$\therefore \ HF^-(Y, s_i) = \mathbb{F}[\mathcal{U}] \Rightarrow HF^-(Y) = \left(\mathbb{F}[\mathcal{U}]\right)^{|\mathrm{Spin}^c Y|}$

$\Rightarrow$ Y is an L-space

$\rightsquigarrow$ All lens spaces are L-spaces.