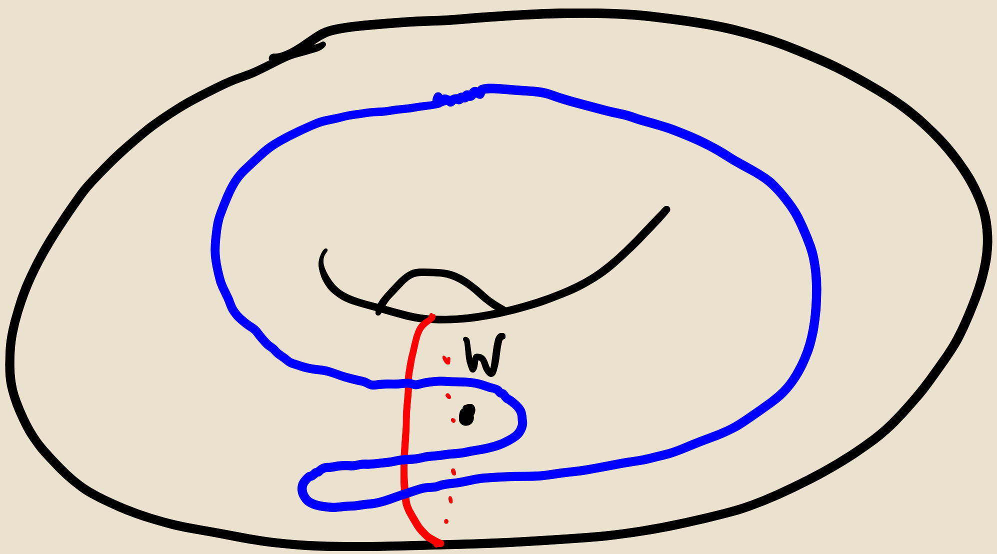
Knot Floer Homology and L-space Surgeries

Jacob Caudell

Recall input for CF°:



Ozsváth-Szabó/
Rasmussen:

As $\mathbb{F}[U]$-modules,

$$CFK^\circ(\Sigma, \vec{\alpha}, \vec{\beta}, w, z) \cong CF^\circ(\Sigma, \vec{\alpha}, \vec{\beta}, w),$$

but the CFK° differential also keeps track of z:

e.g. for $\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$,

$$\hat{\partial}_K \vec{x} = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\vec{x}, \vec{y}) \\ \mu = 1,\, n_w = 0,\, n_z = 0}} \#\left(\widehat{\mathcal{M}}(\phi)\right) \cdot \vec{y}$$

Thm (Oz-Sz, Ras):

$H_*((CFK^\circ(\Sigma,\vec{\alpha},\vec{\beta},z,w),\partial^\circ)$ is an invariant of the knot $K\subset Y$ presented by the doubly pointed Heegaard diagram.

Then

$$H_*((CFK^\circ(\Sigma,\vec{\alpha},\vec{\beta},z,w),\partial_K^\circ) =: HFK^\circ(Y,K)$$

There are two gradings $M, A : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$

<u>Relative definitions</u>: For $\phi \in \pi_2(\vec{x}, \vec{y})$

Maslov (Homological) grading:

$$M(\vec{x}) - M(\vec{y}) = \mu(\phi) - 2n_w(\phi)$$

Alexander grading:

$$A(\vec{x}) - A(\vec{y}) = n_z(\phi) - n_w(\phi)$$

In particular, $\hat{\partial}_z$ lowers M by 1 and preserves A,
$\Rightarrow A$ descends to $\widehat{HFK}(Y,K) \to \mathbb{Z}$

Thm (Oz-Sz, Ras)

There are normalizations of $M, A : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$ such that

$$\sum_{\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} (-1)^{M(\vec{x})} t^{A(\vec{x})} = \Delta_K(t)$$

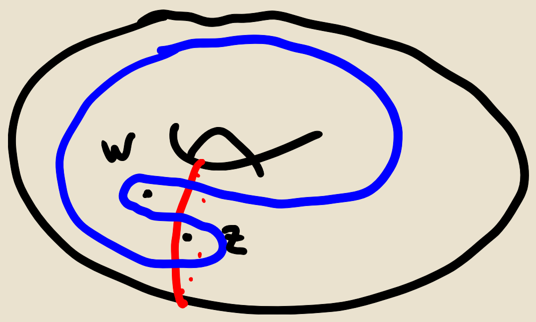
(Alexander polynomial)

Hence, we always choose these normalizations.

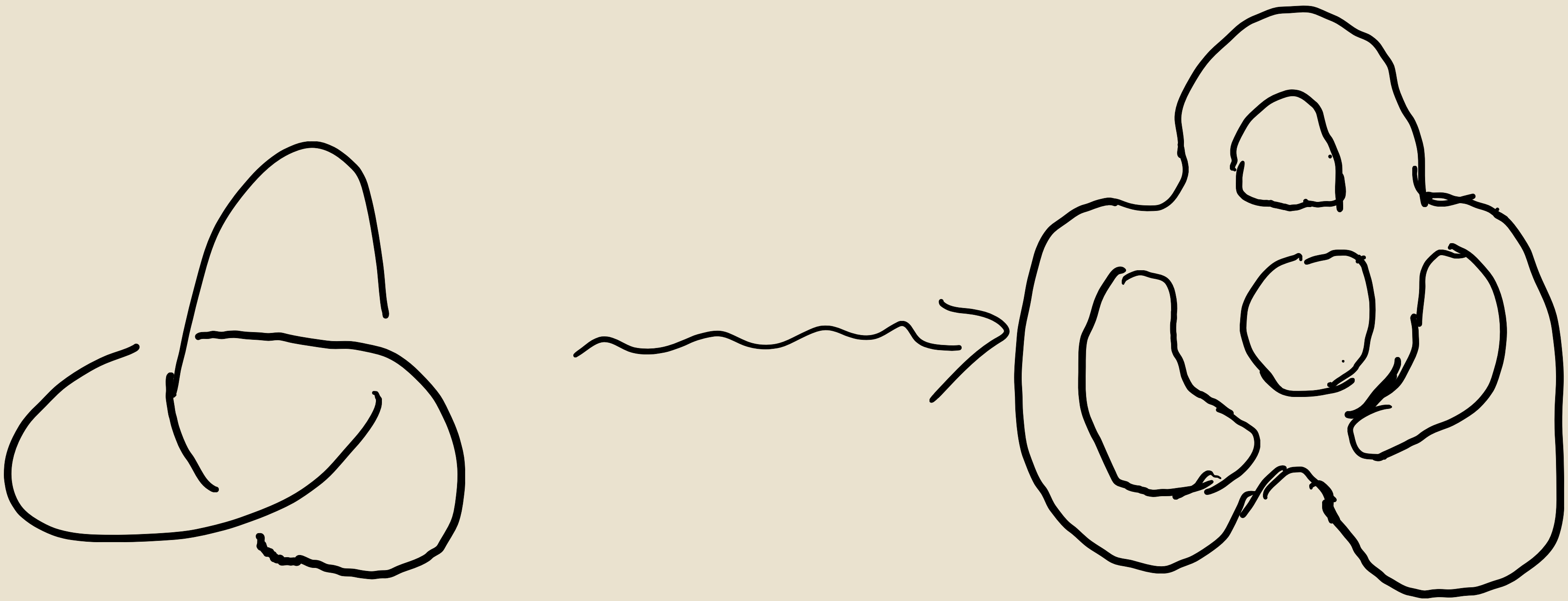
Knot floer homology:

- detects Seifert genus
- detects fiberedness
- categorifies the Alexander Polynomial
- detects unknot, trefoil, and figure 8
- behaves well with Heegaard Floer homology of 3-manifolds

Example: Trefoil



A systematic approach to knots in S^3:



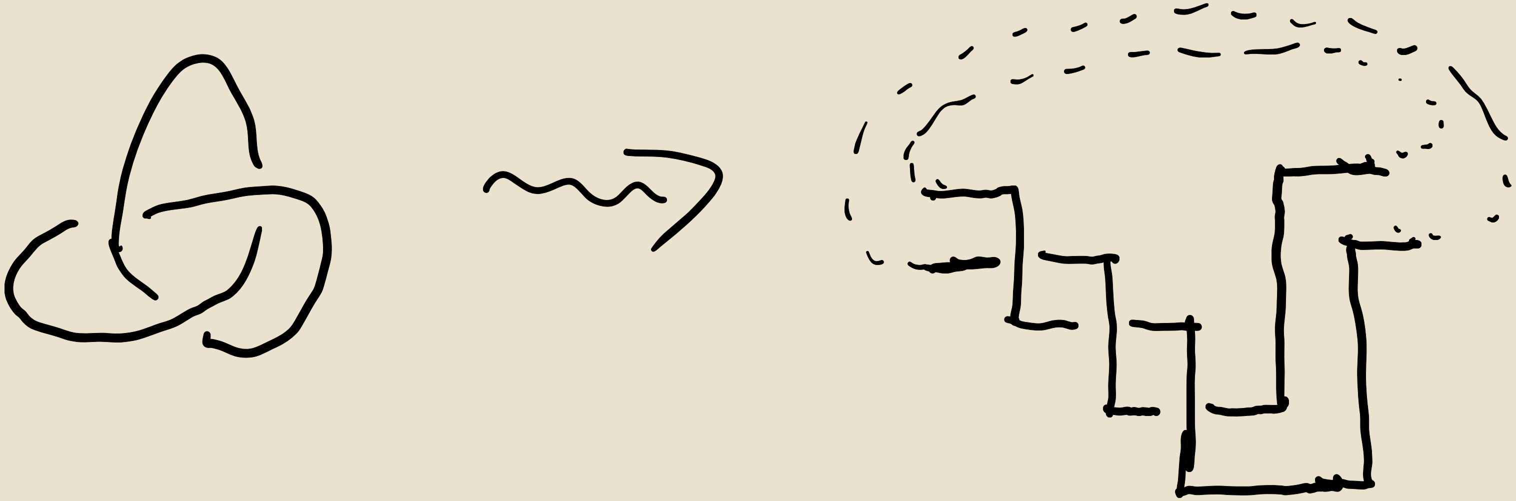
In practice, we cannot easily compute HFK° from Heegaard diagrams of genus ≥ 2.

Some combinatorial workarounds:

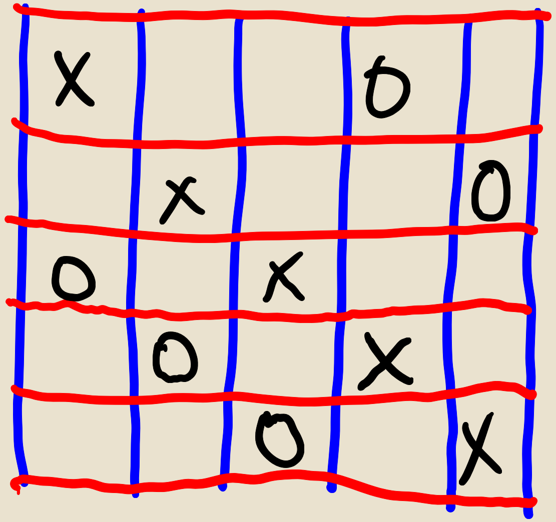
- Nice diagrams (Sarkar-Wang)
- Grid diagrams (Manolescu-Ozsváth-Sarkar)
- Cube of resolutions (Ozsváth-Szabó)
- Bordered Floer homology (Ozsváth-Szabó)

Grid diagrams

Every knot in a Heegaard genus ≤ 1 QHS Y admits a rectilinear projection onto a Heegaard torus of Y:



A grid diagram $\mathbb{G} = (T, X, O)$:



$\rightsquigarrow$ can define a homology theory that agrees with $\widehat{HFK}^{\circ}$ for the underlying knot, but is totally combinatorial

Very computationally expensive, but has nice connections to braid theory and contact geometry.

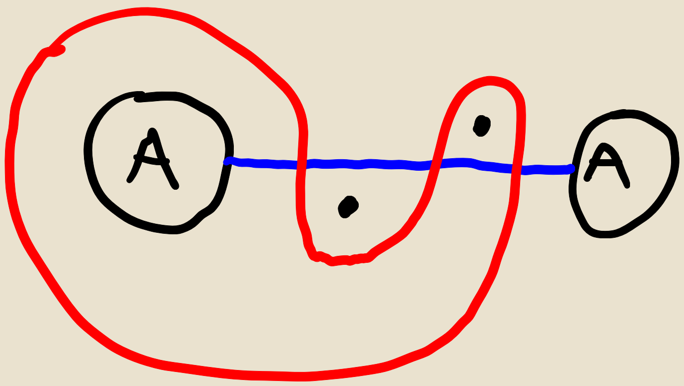
Dehn Surgery

Knots facilitate understanding of 3-mflds.

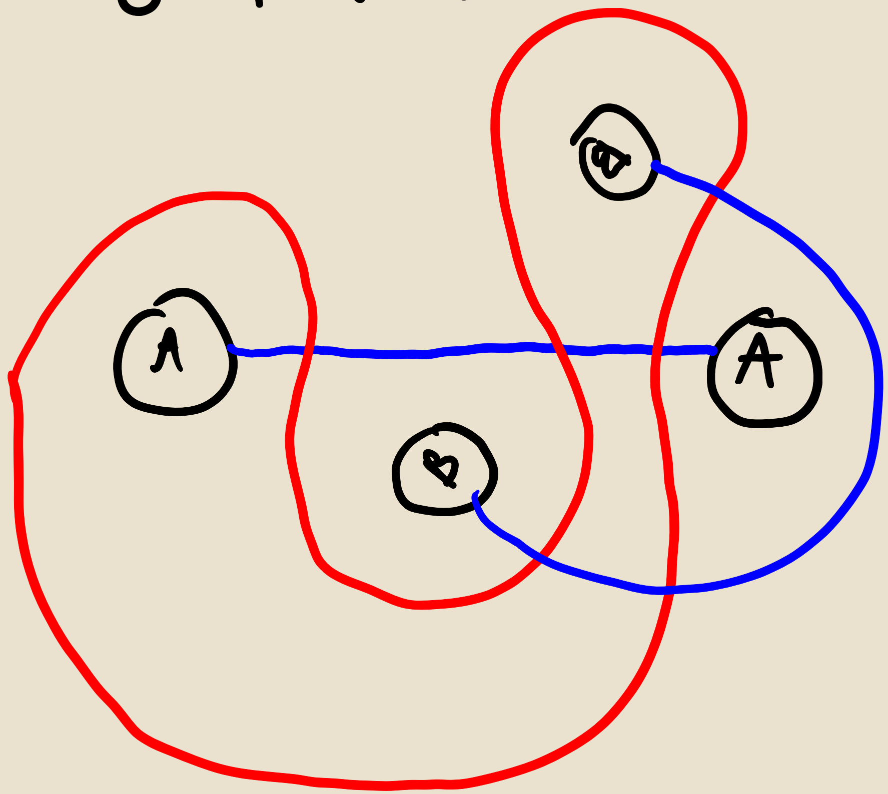
Given a knot $K^1 \hookrightarrow Y^3$, and a slope $\alpha \subset \partial Y^3 - \mathring{\nu}(K)$, $K(\alpha)$ is the 3-manifold obtained by gluing $S^1 \times D^2$ to $Y^3 - \mathring{\nu}(K)$ along their boundaries so that $\{1\} \times \partial(D^2)$ gets identified with α.

E.g.

$T_{2,3} \subset S^3$



Heegaard diagram for $S^3 \setminus \mathring{\nu}(T_{2,3})$



We call a knot $K \subset S^3$ with a non-trivial surgery to an L-space an <u>L-space knot</u>.

E.g. torus knots, knots with surgeries to lens spaces, knots with surgeries to 3-mflds with finite fundamental groups

The figure 8 knot is NOT an L-space knot! More later

L-space knots have some exceptional properties:

For $K \subset S^3$ an L-space knot,

- $HFK^\circ(Y, K)$ is determined by its Alexander polynomial, and moreover if $rk(HFK^\circ(Y,K)) \neq 0$ then $rk(HFK^\circ(Y,K)) = 1$ (Os-Sz '05)
- $S^3 \setminus K$ fibers over S^1 ((Ghiggini '08 genus 1, Ni '07, Juhász '08)
- The set of nontrivial L-space surgery slopes for K is $[2g(K)-1, \infty)$ or $(-\infty, 1-2g(K)]$

Spin^c cobordisms induce homomorphisms on HF°

$$\begin{tikzcd}
(X,\mathfrak{t}) : (Y_0, \mathfrak{t}|_{Y_0}) \arrow[r] \arrow[d, squiggly] & (Y_1, \mathfrak{t}|_{Y_1}) \\
F^\circ_{X;\mathfrak{t}} : HF^\circ(Y_0, \mathfrak{t}|_{Y_0}) \arrow[r] & HF^\circ(Y_1, \mathfrak{t}|_{Y_1})
\end{tikzcd}$$

grading shift, e.g. in HF^+

$$gr(F^+_{X;\mathfrak{t}}(x)) - gr(x) = \frac{c_1(\mathfrak{t})^2 - 2\chi(X) - 3\sigma(X)}{4}$$

then cobordisms $X : Y_0 \to Y_1$

induce maps $F_X = \sum_{t \in Spin^c(X)} F_{X,t}$

(finitely many are non-zero)

Key feature: Surgery exact triangle

Thm: Let $K \subset Y$ be a knot, λ an integer framing of K. Then there is an exact sequence

$$\begin{tikzcd}
 & HF^\circ(K(\lambda)) \arrow[d] \\
HF^\circ(Y) \arrow[ur] & HF^\circ(K(\lambda+1)) \arrow[l]
\end{tikzcd}$$

HF° is functorial: $\partial W^4 = -Y \sqcup Y'$ induces a homomorphism
$F_W : HF^\circ(Y) \to HF^\circ(Y')$

(let λ be a + integer framing)

If $K(\lambda)$ is an L-space, then so is $K(\lambda+1)$!

$$\begin{tikzcd}
 & \widehat{HF}(K(\lambda)) \arrow[d] \\
\widehat{HF}(S^3) \arrow[ur] & \widehat{HF}(K(\lambda+1)) \arrow[l]
\end{tikzcd}$$

look at rk & exactness

$$\begin{tikzcd}
 & \lambda \arrow[d] \\
1 \arrow[ur] & ?? \arrow[l]
\end{tikzcd} \implies \operatorname{rk} \widehat{HF}(K(\lambda+1)) = \lambda+1$$

The key step is showing the map $\widehat{HF}(S^3) \to \widehat{HF}(K(\lambda))$ must be $\equiv 0$.

If this map is 0 and $K(\lambda+1)$ is an L-space, then so is $K(\lambda)$!

This map is 0 iff $\lambda \geq 2g(K) - 1$

The Adjunction Inequality

The map $\widehat{HF}(S^3) \to \widehat{HF}(K(\lambda))$ is induced by the trace cobordism $W_K(\lambda)$ of λ surgery on K. $F_{W_K(\lambda)} = \sum_{t \in spin^c(W_K(\lambda))} F_{W_K(\lambda);t}$

The adjunction inequality states

if $F_{W_K(\lambda);t} \not\equiv 0$ then for every smooth surface $\Sigma \subset W_K(\lambda)$

$$|\langle c_1(t), \Sigma \rangle| + [\Sigma]^2 \leq 2g(\Sigma) - 2$$

Let $\Sigma \subset W_K(\lambda)$ be the surface obtained by capping off a Seifert surface of K with the co-core of λ-surgery.

Then if $\lambda \geq 2g(K) - 1$

$$|\langle c_1(t), \Sigma \rangle| + [\Sigma]^2 \geq \lambda$$

$$= 2g(K) - 1 > 2g(K) - 2$$

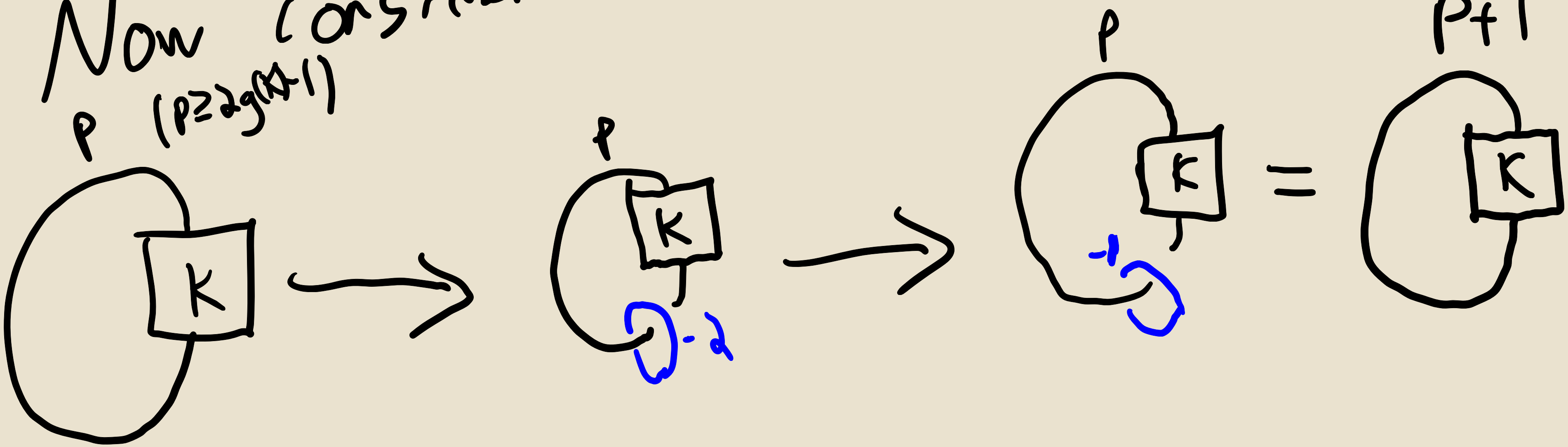
$$= 2g(\Sigma) - 2$$

$$\therefore F_{W_K(\lambda)} = \sum_{t \in Spin^c(W_K(\lambda))} F_{W_K(\lambda), t} \equiv 0$$

Now we know $[2g(K)-1, \infty) \cap \mathbb{Z}$ is contained in the set of L-space slopes of a positive L-space knot K.

Now consider the triple

$p \ (p \geq 2g(K)-1)$



$K(p), K(p+\frac{1}{2}), K(p+1)$ form a triad $M, K(\lambda), K(\lambda+1)$ as in the surgery exact triangle

Now,

$$\left|H_1(K(p+1/2))\right| \leq \operatorname{rk} \widehat{HF}(K(p+1/2)) \leq \operatorname{rk} \widehat{HF}(K(p)) + \operatorname{rk} \widehat{HF}(K(p+1))$$

$$\Rightarrow 2p+1 \leq \operatorname{rk} \widehat{HF}(K(p+1/2)) \leq p + (p+1)$$

$\therefore$ $K(p+1/2)$ is an L-space

This argument serves as a basis for the general statement.

$\widehat{HFK}^+$ and Dehn Surgery

- $\widehat{HFK}^+(Y,K)$ determines $HF^+(K(p/q))$ up to an overall grading shift (K nullhomologous)
- If $K \subset S^3$ is an L-space knot, then for $p/q \geq 0$ and all $|i| \leq p/2q$

$$d(K(p/q),i) - d(U(p/q),i) = -2t_{\lfloor \frac{|i|}{q} \rfloor}(K)$$

while for all $|i| \geq \frac{p}{2q}$ $t_i(K) = 0$